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Abstract

Our problem is to determine and to classify by analytic means all planar shock wave solutions of the hydromagnetic discontinuity relations. The state ahead of the shock (i.e., on the low density side) is assumed to be known; no restriction is placed on the direction of the magnetic field in front. It is shown, apart from certain 'limit' shocks (e.g., pure gas shocks), that the shock velocity and the quantities characterizing the state behind hydromagnetic shocks may be expressed as simple algebraic functions of the discontinuity in the magnetic field across the shock. A natural classification of all hydromagnetic shocks, based on this representation of the state behind the shock, is given. Several useful analytical properties of the various types of hydromagnetic shocks are derived. The results are illustrated graphically for the case of an ideal monatomic gas. The relation between earlier schemes of classification and the present scheme is discussed.

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1. Introduction - Notation, definitions, formulation of the problem

1.1 Introduction

In this report we shall determine and classify, by analytic means, all 'admissible solutions'* of the hydromagnetic discontinuity relations. We shall assume that all discontinuities occur across a plane front which is perpendicular to the x-axis and whose motion, if any, is in the x-direction [see Figure 1].

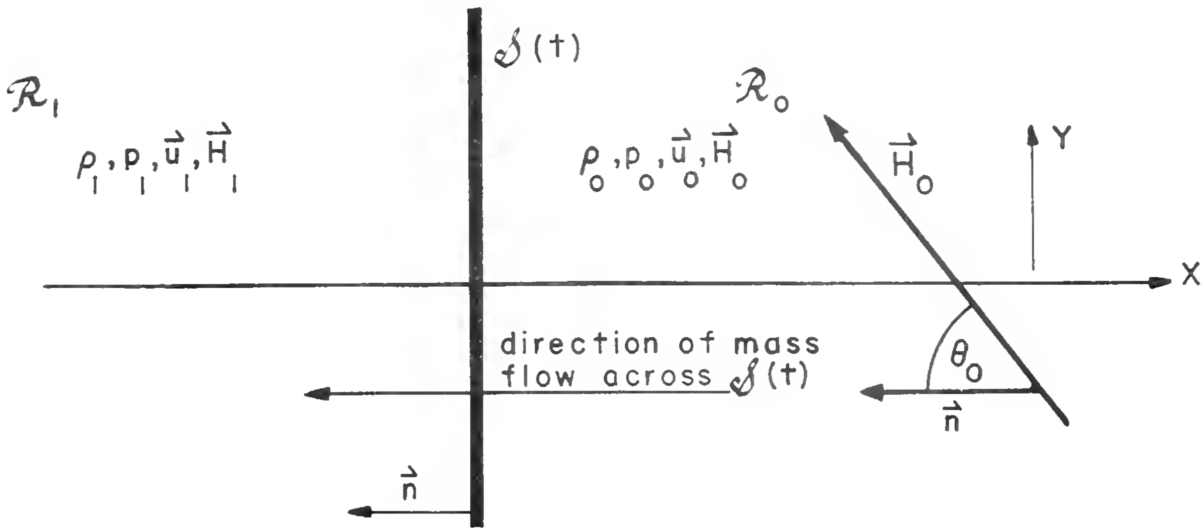

Figure 1: XY-cross section through planar surface of discontinuity $\mathcal{S}(t)$ and the x-axis. $\mathcal{S}(t)$ is parallel to the (y,z)-plane; its motion, if any, is along the x-axis. Assuming the discontinuity to be a shock, the normal to $\mathcal{S}(t)$ is always assumed to be directed toward the region into which the mass flows. This region is labelled with the subscript '1' and referred to as the region 'behind' the shock; the remaining half-space is indexed by 'o' and is referred to as the region in front of the shock. The angle Θ_o is the angle between $\vec{n}$ and $\vec{H}_o$, as shown.

Unless otherwise mentioned, we shall assume, in addition, that the fluid supporting the motion is an ideal polytropic gas. By hydromagnetic discontinuity relations we mean the well-known system of non-relativistic discontinuity relations for a infinitely conducting non-dissipative fluid which was first derived by De Hoffmann and Teller [1] as a limiting case from the corres-

*More precise specifications of the various expressions appearing in 'single quotes' here and below will be given in the next subsection.

ponding relativistic system. Contact discontinuity solutions apart, by an admissible solution we mean any one-parameter family of solutions of these discontinuity relations which, for a fixed value of the parameter (the shock strength parameter), determines the 'state behind the shock', the 'state in front' being assumed given.

Let $\overline{H}_{tr}$ be the component of the magnetic field tangent to the shock front, $\left[\overline{H}_{tr}\right]$ the discontinuity in $\overline{H}_{tr}$ across the shock front and H_o the magnitude of the magnetic field in front of the shock. We shall show that when the ratio $h = \left[H_{tr}\right]/H_o$ (or its negative) is introduced as a shock strength parameter, the state behind compressive magnetic shocks may be represented parametrically in terms of h by means of relatively simple algebraic expressions, valid for all orientations of the magnetic field in front. Apart from its intrinsic interest, this result (and others to be mentioned later) enables us to achieve a more complete and detailed classification of the various shock wave solutions and to penetrate further than has hitherto been possible into questions involving their analytic character.

The plan of our report is as follows: In the next subsection we shall collect the notation definitions and equations to be employed in the remainder of the report and then formulate our problem more precisely. In Section 2 we shall give a detailed summary of our results and discuss their relationship to earlier results. Explicitly, we shall 1) exhibit all admissible solutions of the basic hydromagnetic discontinuity relations, 2) classify these solutions by analytic means, 3) give several useful analytical properties of the solutions, 4) give graphical illustrations of the solutions, and finally 5) relate our scheme of classification and compare (some of) our results with those which have appeared in the literature. The remainder of the report, Sections 3-7, is devoted to giving analytic proofs of the results collected in Section 2.

1.2 Notation, definitions, formulation of the problem

Let $\mathcal{S}(t)$, where t is the time, be a surface of discontinuity which separates flow regions $\mathcal{R}_1$ and $\mathcal{R}_o$ of the fluid. As already mentioned above, we shall suppose that $\mathcal{S}(t)$ is a planar surface; we shall take it parallel to the (y,z)-plane and assume that its motion, if any, is along the x-axis. We shall assume, for the sake of definiteness, that $\mathcal{R}_o$ and $\mathcal{R}_1$ are regions of constant state - i.e., the quantities which describe the flow in these regions are independent of space and time variables [see Figure 1]. The discontinuity is therefore uniform in the sense that its description does not depend on time.

In the following list, all quantities are assumed to be expressed in the (rationalized) MKS Giorgi system of units:

ρ : mass density

$\tau = 1/\rho$: specific volume-i.e., volume per unit mass

p : pressure

e : specific internal energy-i.e., internal energy per unit mass

S : specific entropy - i.e., entropy per unit mass

T : temperature

$\vec{u}$: particle velocity

$\vec{H}$: magnetic intensity

μ : magnetic inductive capacity of free space

$\vec{n}$: unit normal to $\mathcal{S}(t)$ and directed into region $\mathcal{R}_1$

$u_n = \vec{u}\cdot\vec{n}$: normal flow velocity, or particle velocity in the direction of $\vec{n}$

$H_n = \vec{H}\cdot\vec{n}$: component of H in direction of $\vec{n}$

U_n : normal shock velocity - i.e., velocity of shock front in direction of $\vec{n}$

$V_n = u_n - U_n$: normal flow velocity relative to the shock

$m = \rho(u_n - U_n)$: flux of mass through shock front - i.e., mass passing through a unit area in a unit time.

To distinguish between values of a quantity Q in $\mathcal{R}_o$ and $\mathcal{R}_1$ we shall append the subscripts 'o' and '1' and write

$$(1.1) \qquad [Q] = Q_1 - Q_o$$

for the jump in this quantity across the shock; here Q may be a scalar or a vector.

Because of the conservation of mass through the shock front we have

$$(1.2) \qquad m = \rho_1 V_{n,1} = \rho_o V_{n,o} \geq 0.$$

According as m = 0 or m ≠ 0 we shall speak of a <u>shock</u> or a <u>contact discontinuity</u>. Since we are dealing exclusively with one-dimensional motion, it is clear that $\vec{n}$ is directed along the positive or negative x-axis. In the following, <u>assuming that m ≠ 0, we shall employ the convention that $\vec{n}$ points in the direction in which the fluid crosses $\mathcal{S}(t)$</u>. Under these circumstances, m is always positive, a conclusion which we have anticipated in (1.2). The region into which $\vec{n}$ points will be referred to as the region <u>behind</u> the shock; the remaining region will be called the region <u>ahead of</u> or <u>in front of</u> the shock. Since we have labelled the region into which $\vec{n}$ is directed with the subscript '1', it follows that the subscript '1' always refers to the region behind the shock and the subscript 'o' to the region in front of the shock.

In terms of the above notation the basic hydromagnetic shock relations may be expressed as follows [cf. [1]-[4], especially [2]]:

$$(1.3) \qquad \begin{array}{ll}
A_o & [H_n] = 0, \\
A_1 & [m\tau\vec{H}_{tr} - H_n\vec{u}_{tr}] = 0,^{*} \\
A_2 & [m\vec{u} + (p + \mu H^2/2)\vec{n} - \mu H_n\vec{H}] = 0, \\
A_3 & [m] = 0, \text{ or equivalently, } m[\tau] - [u_n] = 0, \\
A_4 & m[u^2/2 + e + \mu H^2\tau/2] + [u_n(p+\mu H^2/2) - \mu H_n\vec{H}\cdot\vec{u}] = 0, \\
A_5 & m[S] \geq 0,
\end{array}$$

---

*$\vec{Q}_{tr}$ is the projection of any vector $\vec{Q}$ on the shock front.

where in $\mathcal{R}_o$ and $\mathcal{R}_1$ the quantities p and e have the form

$$(1.3') \qquad \begin{aligned} p &= A\rho^{\gamma} \\ e &= p\tau/(\gamma-1). \end{aligned}$$

Here, γ is the so-called adiabatic exponent and A is a function of the entropy alone.

A shock wave solution* of A_o - A_4 will be said to be <u>compressive</u> if $[\tau] < 0$, <u>non-compressive</u> if $[\tau] = 0$ and <u>expansive</u> if $[\tau] > 0$. The major portion of our results - those having to do with compressive shocks - will be expressed in the following notation:

$$(1.4) \qquad \begin{aligned} &\text{a)} \quad \bar{\eta} = (\rho_1 - \rho_o)/\rho_o; \qquad \eta = \rho_1/\rho_o = \bar{\eta} + 1, \\ &\text{b)} \quad h = (H_{y,1} - H_{y,o})/H_o, \\ &\text{c)} \quad \bar{Y} = [p]/p_o, \\ &\text{d)} \quad s_j = \gamma p_j/\mu H_j^2, \qquad j = 0,1, \\ &\text{e)} \quad X = \bar{Y}\, s_o/\gamma + h^2/2, \\ &\text{f)}^{**} \quad \sin\theta_j = H_{y,j}/H_j, \qquad -90^\circ \leq \theta_j \leq 90^\circ, \quad j = 0, 1, \end{aligned}$$

and

$$(1.5) \qquad \begin{aligned} b_{n,j} &= \sqrt{\mu H_n^2 \rho_j^{-1}}\ , \\ \left(\frac{c_{s,j}}{b_{n,j}}\right)^2 &= \frac{1 + s_j - \sqrt{(1+s_j)^2 - 4 s_j \cos^2\theta_j}}{2\cos^2\theta_j}\ , \end{aligned}$$

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*Anticipating somewhat, these solutions are not necessarily 'admissible' since they may not satisfy the entropy condition A_5.

**Since none of our formulas will depend in any essential way on the sign of $\cos\theta_j$, $j = 0,1$, there will be no loss of generality in limiting the θ_j- range to the interval $[-90^\circ, 90^\circ]$.

$$\left(\frac{c_{f,j}}{b_{n,j}}\right)^2 = \frac{1+s_j + \sqrt{(1+s_j)^2 - 4s_j\cos^2\theta_j}}{2\cos^2\theta_j}, \quad j = 0,1.$$

Here, $H_j = |\vec{H}_j|$ is the total magnetic field, and θ_j is the angle* between the positive $\vec{n}$-direction and $\vec{H}_j$ in region $\mathcal{R}_j$, $j = 0,1$; $\bar{\eta}$ is the excess density ratio, and $\bar{Y}$, the excess pressure ratio. The dimensionless ratio h will be employed as a measure of the discontinuity in the transverse magnetic field, while the quantity X will be used primarily as a means of simplifying the appearance of the shock relations when the present notation is introduced. The quantity $b_{n,j}$ is the Alfvén or, as we shall sometimes say, 'intermediate' disturbance speed in a direction making an angle $\theta_j = \widehat{\vec{n}\,\vec{H}_j}$ with the magnetic field $\vec{H}_j$. The ratio s_j may be interpreted as the squared ratio of the sound speed

(1.6) $$a_j = \sqrt{\gamma p_j \rho_j^{-1}}$$

to the Alfvén disturbance speed in the direction of $\vec{H}_j$, namely $\sqrt{\mu H_j^2 \rho_j^{-1}}$. The quantities $c_{s,j}$ and $c_{f,j}$ are, in Friedrichs' terminology**, the slow and fast disturbance speeds in $\mathcal{R}_j$. As will be recalled, a_j, $b_{n,j}$ $c_{s,j}$ and $c_{f,j}$ satisfy the relations [cf. [2] p. 12]:

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*Angles measured clockwise from the positive $\vec{n}$-direction to the positive $\vec{H}$-direction are considered positive-otherwise, negative [see Figure 1].

**The existence of three disturbance speeds was discovered by Herlofson [6] and independently by Van de Hulst [7] at about the same time. The speed c_s corresponds to the speed of Van de Hulst's slow mode, c_f to his fast mode and b_n to the Alfvén mode. In the limit of small $s = \gamma p/\mu H_n^2$, Van de Hulst refers to c_s as the retarded sound speed, c_f as the modified Alfvén wave.

$$c_{s,j} \leq b_{n,j} \leq c_{f,j}$$

(1.7)

$$c_{s,j} \leq a_j \leq c_{f,j}, \qquad j = 0,1.$$

By the <u>state</u> Γ_o <u>in front of a shock</u> we shall mean the set $(\rho_o, p_o, \vec{u}_o, \vec{H}_o)$ or any equivalent set - e.g., $\Gamma_o = (p_o, s_o, \sin\theta_o, \vec{u}_o/b_{n,o}, H_n)$. <u>By the state</u> Γ_1 <u>behind a shock</u> we shall mean the set $\Gamma_1 = (\rho_1, p_1, \vec{u}_1, \vec{H}_1; U_n)$ or any equivalent set - e.g., $\Gamma_1 = (\bar{\eta}_1, \bar{Y}_1, \vec{u}_1/b_{n,1}, h; m)$. The state in front of a shock will always be assumed known at the outset.

Consider now the basic discontinuity relations (1.3) A_o-A_4. Since H_n is continuous across $\mathcal{S}(t)$ and since Γ_o is known, A_1-A_4 constitute seven conditions on the eight unknowns $\Gamma_1 = (\rho_1, p_1, \vec{u}_1, \vec{H}_{tr,1}, m)$. Clearly, A_o-A_4 determine, in general, a one-parameter family of states behind the shock. Let ε, $0 < \varepsilon \leq \varepsilon_1$ denote the generic shock strength parameter*; ε may be any scalar quantity - e.g., $\bar{\eta}, \bar{Y}, h$ etc. Then any determination of the state,

$$(1.8) \qquad \Gamma_1 = \Gamma_1(\varepsilon, \Gamma_o), \quad 0 < \varepsilon \leq \varepsilon_1,$$

behind the shock which a) depends continuously on the shock strength parameter ε, $0 \leq \varepsilon \leq \varepsilon_1$, and the state in front, and which b) satisfies the entropy condition A_5 will be called an <u>admissible solution</u> of the discontinuity relations.

Our problem is 1) to determine all admissible solutions of the discontinuity relations, 2) to ascertain some of the general features of these solutions and 3) to classify admissible solutions in a natural way by analytic means.

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*Our choice of parameters will be such that $\varepsilon = 0$ corresponds to a continuous transition across $\mathcal{S}(t)$. Thus, when we speak of a shock we must require that $\varepsilon > 0$; ε_1 may be infinite, in which case '$\leq$' must be replaced by '$<$' in (1.8).

2. Survey of results

2.1 Preliminary remarks

Admissible solutions of the discontinuity relations (1.3) may be classified in accordance with the following exhaustive scheme. First, we distinguish between two families of solutions; contact discontinuity solutions ($m=0$) and shock wave solutions ($m\neq 0$). Second, shock wave solutions are divided into three mutually exclusive classes: compressive shocks ($[\tau] \equiv [\rho^{-1}] < 0$), non-compressive shocks ($[\tau] = 0$) and expansive shocks ($[\tau] > 0$). Employing a result to be proved in Section 3, that only compressive and non-compressive shocks are admissible*, we eliminate expansive shocks from consideration. As an aside, it may be mentioned that this result is equivalent to one already given by Friedrichs [2] which in essence amounts to this: A shock is compressive if and only if the entropy behind the shock exceeds the entropy ahead [cf. [2] p. 26]. In addition, we shall show in Section 3 that $[\tau] \leq 0$ implies $[p] \geq 0$ and conversely**. Returning to our classification, we subdivide the class of all compressive shocks as follows:

Fast magnetic shocks (M_f shocks): $h \sin \theta_0 > 0$; $0 < \theta_0 < 90°$,

Slow magnetic shocks (M_s shocks): $h \sin \theta_0 < 0$; $0 < \theta_0 < 90°$,

(2.1) Limit shocks : a) Fast $0°$-limit shocks
b) Slow $0°$-limit shocks
c) Fast $90°$-limit shocks
d) Slow $90°$ - limit***

Here we have further restricted the range of θ_0 to $0 \leq \theta_0 \leq 90$. That this restriction involves no essential loss of generality, will be clear later.

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*Expansive shocks violate the entropy condition ($1.3 - A_5$).

**More accurately, $[\tau] < 0$ implies $[p] > 0$ and $[\tau] = 0$ implies $[p] = 0$.

***This limit is a contact discontinuity [see subsection 2.6.5 below].

The adjective 'magnetic' is used to characterize all compressive shocks in which $h \neq 0$. The use of 'fast' and 'slow' will be justified later. From the definition of h [see (1.4 -b)] it follows that $H_{y,1} > H_{y,o}$ in M_f shocks and $H_{y,1} < H_{y,o}$ in M_s shocks*. Limit shocks, as the name suggests, are shocks which as limits of $M_f(\theta_o)$ and $M_s(\theta_o)$ shocks as θ_o approaches 0^o or 90^o.

Anticipating somewhat, this classification may be further refined by dividing M_f and M_s shocks into two subclasses $M_f^{(1)}, M_s^{(1)}$ and $M_f^{(2)}$, $M_s^{(2)}$ say, according as the excess density ratio, excess pressure ratio, etc. are singled-valued functions of h for all allowable values of h or are double-valued functions of h in part of the allowable range. We shall see presently how this division as to 'type' is effected analytically. Here we would like merely to mention that this refinement clarifies the relationship of M_f and M_s shocks to their 0^o-limits. As we shall see later, $M_f^{(1)}$ and $M_f^{(2)}$ shocks have 0^o-limits which are essentially different; the same applies to the 0^o-limits of $M_s^{(1)}$, $M_s^{(2)}$ shocks.

Another important means of classifying hydromagnetic shocks involves the relation of $V_{n,1}$, the normal flow velocity (in the region behind the shock) relative to the shock, to $b_{n,1}$, the intermediate disturbance speed in the region behind the shock. In this scheme, a shock is called <u>fast</u> if

(2.2) $\quad V_{n,1} \gtrsim b_{n,1}, \qquad 0 < \varepsilon \leq \varepsilon_1,$

<u>intermediate</u> if

(2.3) $\quad V_{n,1} = b_{n,1}, \qquad 0 < \varepsilon \lesssim \varepsilon_1,$

and <u>slow</u> if

*We have assumed implicitly that the transverse magnetic field ahead of and behind a magnetic shock is parallel to the y-axis; we shall show in Section 4 that this assumption entails no loss of generality.

(2.4) $\qquad V_{n,1} \leq b_{n,1}\,, \qquad\qquad 0 < \varepsilon \leq \varepsilon_1;$

in (2.2) and (2.4) we require that the inequality holds in at least some proper subinterval of $0 < \varepsilon \leq \varepsilon_1$. Fast and slow magnetic shocks will turn out to be fast and slow in this sense.

Our scheme of classification is now essentially complete. That it is a natural one will become apparent as we proceed.

In the following, we shall exhibit all admissible solutions of the basic discontinuity relations and carry out the remainder of the plan detailed at the end of Subsection 1.2. Contact discontinuity solutions, non-compressive shock solutions (which turn out to be 'transverse' shocks*), the so-called parallel shock solutions (characterized by the condition $h = 0$) and perpendicular shock solutions (characterized by the condition $H_n = 0$) are of course well known; these will therefore be accorded a summary treatment.

2.2 Contact discontinuities

Setting $m = 0$ in (1.3) we find that <u>if $H_n \neq 0$</u> then

(2.5) $\qquad [\vec{u}] = 0,\ [\vec{H}] = 0,\ [p] = 0;\ [\tau],\ [S]$ arbitrary,

while <u>if $H_n = 0$</u> then

(2.6) $\qquad [u_n] = 0,\ [p + \mu H_{tr}^2/2] = 0;\ \ [\vec{u}_{tr}],\ [H_{tr}],\ [\tau],\ [S]$ arbitrary,

[cf. [2] p. 36]. We conclude from (2.5) and (2.6) that a shear flow discontinuity is an inadmissible hydromagnetic motion unless the magnetic field normal to the surface of discontinuity $\mathcal{S}(t)$ vanishes. This interesting

*This is Friedrichs'[2] terminology. De Hoffmann and Teller[1] call these shocks 'symmetrical' shocks.

fact was first observed by Friedrichs[2] and served as the basis of an investigation by him and later by Bazer[8] of the motion which 'resolves' an initial shear flow discontinuity ($H_n \neq 0$).

2.3 Non-compressive shocks ($m \neq 0$, $[\tau] = 0$)-Transverse shocks

Setting $[\tau]$ equal to zero in (1.3) we find that non-trivial solutions exist if, and only if, [cf. [1] - [4]]

$$(2.7) \qquad m^2 = \mu H_n^2 \tau^{-1}$$

and that under these circumstances

$$[u_n] = 0, \quad [p] = 0,$$

$$(2.8) \qquad [\vec{u}_{tr}] = \sqrt{\mu\tau} \ [\vec{H}_{tr}], \qquad H^2_{tr,1} = H^2_{tr,0},$$

$$[S] = 0.$$

These shocks are the well-known transverse shocks, the word 'transverse' being used to emphasize that the jump in the flow is along $\mathcal{S}(t)$*. Transverse shocks may be termed intermediate in the sense of (2.3); for, using (2.7) and (2.8) we have

$$(2.9) \qquad V_{n,1} = V_{n,o} = b_{n,1} = b_{n,o} \ .$$

Note that $\vec{H}_{tr,1}$ behind the shock need not be collinear with the transverse magnetic field in front; all that is required is that the magnitude of $\vec{H}_{tr}$

*As seen from suitable frame of reference, the flow on both sides will be parallel to the shock front.

be continuous across the shock. The magnetic field therefore rotates in the plane of the shock through some arbitrary angle. [S] vanishes because [τ] vanishes [see Section 3].

2.4 Fast magnetic shocks ($m \neq 0$, $h \sin \theta_o > 0$, $\theta_o \neq 0^o, 90^o$)

Fast magnetic shocks or M_f shocks are a subclass of compressive shocks. The state behind these shocks is determined by the following relations:

$$M_{f,1} \quad \bar{\eta}_f = h_f \left\{ \frac{\left(-\frac{\gamma}{2}h_f \sin\theta_o - (1-s_o)\right) \pm \sqrt{R(h_f)}}{2s_o \sin\theta_o - (\gamma-1)h_f} \right\},$$

$$M_{f,2} \quad \bar{Y}_f = \frac{\gamma}{s_o}\left\{-\frac{h_f^2}{2} + h_f\left[\frac{\left(\frac{\gamma}{2}h_f \sin\theta_o - (1-s_o)\right) \pm \sqrt{R(h_f)}}{2\sin\theta_o - (\gamma-1)h_f}\right]\right\},$$

$$= \frac{\gamma}{s_o}\left\{-\frac{h_f^2}{2} + h_f\left[\frac{\bar{\eta}_f/h_f - \sin\theta_o}{1-(\bar{\eta}_f/h_f)\sin\theta_o}\right]\right\},$$

(2.10)

$$M_{f,3} \quad \frac{V^f_{n,1}}{b^f_{n,1}} = \eta_f^{-1/2}\frac{V^f_{n,o}}{b_{n,o}} = \frac{m_f}{\rho_1^f b^f_{n,1}} = \eta_f^{-1/2}\frac{m_f}{\rho_o b_{n,0}} = \frac{1}{\left[1-(\bar{\eta}_f/h_f)\sin\theta_o\right]^{1/2}},$$

$$M_{f,4} \quad \frac{[u_n]_f}{b^f_{n,1}} = -\bar{\eta}_f \frac{V^f_{n,1}}{b^f_{n,1}} = -\frac{\bar{\eta}_f}{\left[1-\bar{\eta}_f/h_f \sin\theta_o\right]^{1/2}},$$

$$M_{f,5} \quad \frac{[u_y]_f}{b^f_{n,1}} = \frac{b^f_{n,1}}{V^f_{n,1}}\frac{h_f}{\cos\theta_o} = \frac{h_f}{\cos\theta_o}\left[1-(\bar{\eta}_f/h_f)\sin\theta_o\right]^{1/2},$$

where

(2.11) $$R(h_f) = h_f^2\left\{\frac{\gamma^2 \sin^2\theta_o}{4} - (\gamma-1)\right\} + h_f \sin\theta_o (2-\gamma)(1+s_o) + 4s_o \sin^2\theta_o + (1-s_o)^2,$$

and where 'f' is used to identify M_f shock quantities.*

The values of the quantities in the left-hand sides of equations (2.10) are completely determined as soon as the state in front is prescribed and the value of h_f is specified; $\bar{\eta}_f$, $\bar{\eta}_f/h_f$, $\bar{Y}_f$, $\eta_f(=\bar{\eta}_f+1)$ may be calculated from $M_{f,1}$ and $M_{f,2}$; $V^f_{n,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, $[u_n]_f/b^f_{n,1}$ and $[u_y]_f/b^f_{n,1}$ may be calculated from the knowledge of how $\bar{\eta}_f$, $\bar{\eta}_f/h_f$ and η_f depend on h_f. These quantities may, of course, be expressed directly in terms of h_f; however, there is no real gain in doing this.

The two 'types' of M_f shocks - i.e., the $M_f^{(1)}$ and $M_f^{(2)}$ shocks mentioned above-are defined by the following relations:

(2.12) $$M_f^{(1)}:\quad s_o \geq 1-\gamma\sin^2\theta_o/(\gamma-1),$$

(2.13) $$M_f^{(2)}:\quad s_o < 1-\gamma\sin^2\theta_o/(\gamma-1).$$

In $M_f^{(1)}$ shocks only the 'plus' branches in equations $M_{f,1}$ and $M_{f,2}$ above are admissible; the negative branch leads ultimately to solutions which violate the entropy condition. To be sure, the plus branch is not admissible for all values of h_f. The appropriate h_f range is:

(2.14) $$0 < h_f \leq \hat{h}_f = \frac{2}{\gamma-1}\sin\theta_o.$$

In this range, $\bar{\eta}_f$, $\bar{Y}_f$, $V^f_{n,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, m_f and $[S]_f$ vary in the same sense as h_f - i.e., they increase or decrease according as h_f increases or decreases.**

*Note that $M_{f,1}$ - $M_{f,4}$ remain unchanged when θ_o is replaced by $-\theta_o$, and h_f by $-h_f$, while $M_{f,5}$ merely changes sign; the restriction of θ_o to the range $0 < \theta_o < 90^\circ$ therefore involves no essential loss of generality [see bottom of p. 8].

**In the following we shall limit ourselves to discussing the qualitative behavior of $\bar{\eta}_f$, $\bar{Y}_f$ m_f and $[S]_f$ (and some directly related quantities). The behavior of $[\vec{u}]$ may, of course, be obtained from the knowledge of how these quantities vary; but, for the sake of brevity, we shall not discuss the behavior of this quantity.

As h_f increases from zero to $\hat{h}_f$, $\bar{\eta}_f$ increases from zero to $2/(\gamma-1)$, the familiar gas-dynamical limit. At the same time, the variables $\bar{Y}_f$, $V^f_{n,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, and $[S]_f$ increase from zero to infinity, becoming asymptotic to the vertical line of $h_f = 2\sin\theta_o/(\gamma-1)$. The dependence of these quantities on h_f is clearly illustrated in Figure 2*, where we have plotted $\bar{\eta}_f$, $\bar{Y}_f$, etc. versus h_f for several values of the parameter $\theta_o > 0$. In this figure, $s_o = 1$; thus all curves are of a type 1. In Figure 3, where we have plotted a similar family of curves corresponding to $s_o = 1/16$, only those curves indexed by values of $\theta_o \geq 37^o46'$ are of type 1.

In $M_f^{(2)}$ shocks both branches of $M_{f,1}$ and $M_{f,2}$ must be employed. The plus branch yields part of the complete admissible solution when h_f is in the range

$$0 < h_f \leq \hat{\hat{h}}_f ,$$

where

$$\hat{\hat{h}}_f = \frac{\sin\theta_o(2-\gamma)(1+s_o) + 2\cos\theta_o\sqrt{(\gamma-1)(1-s_o)^2 + s_o\gamma^2\sin^2\theta_o}}{2(\gamma-1) - (\gamma^2\sin^2\theta_o/2)} \tag{2.15}$$

can be shown to exceed $\hat{h}_f$ [see (2.14) above].** The remaining part of the

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*In all our graphical illustrations we have taken γ to be 5/3. Figures 2-7 will be found at the end of this section.

**It is easily verified that $\bar{\eta}_f$ and $\bar{Y}_f$ remain finite at $h_f = \hat{h}_f$ on the plus branch of $M_f^{(2)}$ shocks.

solution is furnished by the negative branch which is admissible in the interval

(2.16) $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$,

and which joins the plus branch smoothly at $h_f = \hat{\hat{h}}_f$; at this point, $R(h_f)$ [see (2.11)] vanishes. On the plus branch, $\bar{\eta}_f$, $\bar{Y}_f$, $V^f_{n,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, m_f and $[S]_f$ vary directly with h_f; as h_f increases from zero to $\hat{\hat{h}}_f$, these quantities increase monotonically from zero and assume finite values at $h_f = \hat{\hat{h}}_f$ at which point their h_f-derivatives become infinite. On the negative branch, the quantities $\bar{\eta}_f$, $\bar{Y}_f$, $V^f_{n,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, m_f and $[S]_f$ vary in a sense opposite to that of h_f. As h_f decreases from $\hat{\hat{h}}_f$ to $\hat{h}_f$, $\bar{\eta}_f$ increases monotonically to the limiting value $2/(\gamma-1)$, while the remaining variables increase monotonically to $+\infty$, approaching the line $h_f = \hat{h}_f$ asymptotically. The h_f-derivatives of all quantities are infinite at $h_f = \hat{\hat{h}}_f$; the joining of the negative branch with the (lower) positive branch is therefore smooth. These features are clearly brought out in the type 2-curves of Figure 3 - i.e., those corresponding to values of $\theta_o < 37^o 46'$.

In all M_f shocks, whether of type 1 or type 2 it can be shown that $\bar{Y}_f$, $V^f_{n,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, m_f <u>and</u> $[S]_f$ <u>vary in the same sense as</u> $\bar{\eta}_f$. This fact is illustrated for $\bar{Y}_f$ and $\bar{\eta}_f$ in Figure 4 where we have plotted $\bar{Y}_f$ against $\bar{\eta}_f$. In this graph the upper curve corresponding to $\sin \theta_o = 0.1$ is a curve of type 2; yet $\bar{Y}_f$ increases with increasing $\bar{\eta}_f$.

We have already remarked that M_f shocks are fast in the sense of (2.2). The proof of this remark follows immediately from (2.10-$M_{f,3}$); for, since $\bar{\eta}_f$ and h_f are positive, the last expression is clearly greater than unity. We therefore have

$$V^f_{n,1} > b_{n,1}, \quad 0 < \varepsilon \leq \varepsilon_1 ,$$

where it will be recalled, ε is the generic shock strength parameter.

It can also be shown that

(2.17) $$V^f_{n,1} < c_{f,1}, \quad 0 < \varepsilon \leq \varepsilon_1,$$

and

(2.18) $$V^f_{n,o} > c_{f,o}, \quad 0 < \varepsilon \leq \varepsilon_1 .$$

Thus, in $\underline{M_f}$-shocks the normal flow velocity relative to the shock is sub-fast behind, and super-fast in front. This result is analogous to the well-known gas dynamical result; namely, the normal flow velocity relative to the shock is subsonic behind, and supersonic in front.

In the weak shock limit we find that

(2.19) $$\lim_{\varepsilon \to o} V^f_{n,1} = \lim_{\varepsilon \to o} V^f_{n,o} = c_{f,o}.$$

The state behind weak shocks is determined by the following set of equations:*

$$\Delta M_{f,1} \qquad h_f = \frac{\sin \theta_o}{1 - r_f} \bar{\eta}_f + \cdots ,$$

$$\Delta M_{f,2} \qquad \bar{Y}_f = \gamma \, \bar{\eta}_f + \cdots ,$$

(2.20) $$\Delta M_{f,3} \qquad m_f \tau_1 = V_{n,1} = V_{n,o} = c_{f,o} + \cdots ,$$

$$\Delta M_{f,4} \qquad [u_n]_f = -c_{f,o} \bar{\eta}_f + \cdots ,$$

$$\Delta M_{f,5} \qquad [u_y]_f = \frac{r_f c_{f,o} \tan \theta_o}{1-r_f} \bar{\eta}_f + \cdots ,$$

---

*If $\bar{\eta}_f$ is identified with $k(\rho_o c^2_{f,o} - \mu H^2_n)$, where k is an arbitrary constant, and this quantity is substituted for $\bar{\eta}_f$ in (2.20), then the resulting formulas may easily be shown to agree with fast small disturbance formulas derived by Friedrichs in [2] [see p. 13].

where

$$(2.21) \qquad r_f = \frac{b^2_{n,o}}{c^2_{f,o}} < 1.$$

In addition, we find that

$$(2.22) \qquad [S]_f = c_v \frac{\gamma(\gamma-1)}{4} \left\{ \frac{\gamma+1}{3} + \frac{\sin^2\theta_o}{(1-r_f)^2 s_o} \right\} \bar{\eta}_f^{\,3} + O(\bar{\eta}_f^{\,4}).$$

Here, c_v is the specific heat at constant volume. Since $\bar{Y}_f$, h_f, $[u_n]_f$, $[u_y]_f$, etc. depend linearly on each other we conclude that the increase of entropy across an M_f shock is of third order in the shock strength.

Finally, we remark that when $s_o < 1$ and $h_f = 2(1-s_o)/\gamma$, the values of $\bar{\eta}_f$, $\bar{Y}_f$ and $[S]_f$ are independent of $\sin\theta_o$. Explicitly, we find that

$$(2.23) \qquad \begin{aligned} \bar{\eta}_f &= \frac{2}{\gamma}(1-s_o) , \\ \bar{Y}_f &= \frac{2}{\gamma s_o}(1-s_o)(\gamma - 1 + s_o), \\ [S]_f &= c_v \log\left[\frac{2(1-s_o)(\gamma-1+s_o) + \gamma s_o}{\gamma s_o}\right] - c_v\gamma \log\left[\frac{2(1-s_o)+\gamma}{\gamma}\right] \end{aligned}$$

when

$$(2.24) \qquad h_f = 2(1-s_o)/\gamma, \qquad s_o < 1.$$

This property of $M_f^{(2)}$ shocks is illustrated in Figures 3 and 4.

2.5 Slow magnetic shocks ($m \neq 0$, $h \sin\theta_o \leq 0$, $\theta_o \neq 0^\circ, 90^\circ$)

Slow magnetic shocks - that is, M_s shocks - are a subclass of compressive shocks. The state behind these shocks is determined by the following relations:

$$M_{s,1} \quad \bar{\eta}_s = h_s \left\{ \frac{((1-s_o)-\gamma h_s \sin\theta_o/2) \pm \sqrt{\hat{R}(h_s)}}{(\gamma-1)h_s + 2s_o \sin\theta_o} \right\},$$

$$M_{s,2} \quad \bar{Y}_s = \frac{\gamma}{s_o} \left\{ -\frac{(h_s)^2}{2} + h_s \left[\frac{((1-s_o) + \frac{\gamma}{2} h_s \sin\theta_o \pm \sqrt{\hat{R}(h_s)}}{2\sin\theta_o + (\gamma-1)h_s} \right] \right\},$$

(2.25)

$$M_{s,3} \quad \frac{V^s_{n,1}}{b^s_{n,1}} = \frac{1}{\eta_s^{1/2}} \frac{V^s_{n,o}}{b_{n,o}} = \frac{m_s}{\rho^s_1 b^s_{n,1}} = \frac{1}{\eta_s^{1/2}} \frac{m_s}{\rho_o b_{n,o}} = \frac{1}{\left[1+(\bar{\eta}_s/h_s)\sin\theta_o\right]^{1/2}},$$

$$M_{s,4} \quad \frac{[u_n]_s}{b^s_{n,1}} = -\bar{\eta}_s \frac{V^s_{n,1}}{b^s_{n,1}} = -\frac{\bar{\eta}_s}{\left[1+(\bar{\eta}_s/h_s)\sin\theta_o\right]^{1/2}},$$

$$M_{s,5} \quad \frac{[u_y]_s}{b^s_{n,1}} = -\frac{b^s_{n,1}}{V^s_{n,1}} \frac{h_s}{\cos\theta_o} = -\frac{h_s}{\cos\theta_o} \left[1+(\bar{\eta}_f/h_f)\sin\theta_o\right]^{1/2},$$

where

$$\hat{R}(h_s) = (h_s)^2 \left\{ \frac{\gamma^2 \sin^2\theta_o}{4} - (\gamma-1) \right\} - h_s \sin\theta_o (2-\gamma)(1+s_o) \tag{2.26}$$

$$+ 4s_o \sin^2\theta_o + (1-s_o)^2.$$

Here, we have employed the subscript 's' to identify M_s shock quantities and have introduced the quantity

$$h_s = -h = (H_{y,o} - H_{y,1})/H_o \tag{2.27}$$

as a shock strength parameter. The system (2.25) may be obtained from (2.10) on substituting $(-h_s)$ for h_f. Since we are restricting ourselves to positive values of θ_o (see bottom of p. 8), it follows from the definition of M_s shocks that

(2.28) $\quad h_s > 0.$

The two 'types' of M_s shocks are defined by the following relations:

(2.29) $\quad M_s^{(1)}: \quad s_o \geq 1 - \gamma \sin^2\theta_o,$

(2.30) $\quad M_s^{(2)}: \quad s_o < 1 - \gamma \sin^2\theta_o.$

In $M_s^{(1)}$ shocks only the 'plus' branches in equations $M_{s,1}$ and $M_{s,2}$ above are admissible and then only when h_s is in the range

(2.31) $\quad 0 < h_s \leq \hat{h}_s = 2 \sin \theta_o.$

In contrast to the behavior of the corresponding quantities in M_f shocks, the quantities $\bar{\eta}_s$, $\bar{Y}_s$, $V^s_{n,1}/b^s_{n,1}$, $V^s_{n,o}/b_{n,o}$, m_s and $[S]_s$ do not depend monotonically on h_s. Consider, for example, the variation of $\bar{\eta}_s$, $\bar{Y}_s$ and $[S]_s$ with h_s. As h_s increases from zero to $\hat{h}_s$, these quantities reach finite maxima at suitable (in general not the same*) interior points and finally return to zero at $h_s = \hat{h}_s$. This behavior is illustrated in Figures 5(a), (b) and (e) where we have plotted $\bar{Y}_s$, $\bar{\eta}_s$ and $[S]_s/c_v$ versus h_s for several values of $\theta_o \geq 0$; since $s_o = 1$, all curves are of type 1. It should be observed that the magnitude of the transverse magnetic field behind $M_s^{(1)}$ shocks is less than or equal to that ahead of $M_s^{(1)}$ shocks.

In $M_s^{(2)}$ shocks both the plus and minus branches are admissible in suitable h_s-ranges. The plus branch yields part of the complete admissible solutions when h_s is in the range

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*It can, in fact, be shown that the value of h_s at which $[S]_s$ is maximum exceeds the value of h_s at which $\bar{Y}_s$ is maximum which in turn exceeds the value at which $\bar{\eta}_s$ is maximum.

(2.32) $\quad 0 < h_s \leq \hat{\hat{h}}_s,$

where

(2.33) $$\hat{\hat{h}}_s = \frac{-\sin\theta_o(2-\gamma)(1+s_o) + 2\cos\theta_o\sqrt{(\gamma-1)(1-s_o)^2 + s_o\gamma^2\sin^2\theta_o}}{2(\gamma-1) - (\gamma^2\sin^2\theta_o/2)}$$

can be shown to exceed $\hat{h}_s$ (see (2.31) above). The remaining part of the solution is furnished by the minus branch which is admissible in the interval

(2.34) $\quad \hat{h}_s \leq h_s \leq \hat{\hat{h}}_s$

and which joins the plus branch at $h_s = \hat{\hat{h}}_s$; at this point $\hat{R}(h_s)$ (see 2.26) vanishes. Thus Γ_1, the state behind, is a double-valued function of h_s in the interval $\hat{h}_s \leq h_s \leq \hat{\hat{h}}_s$. On the plus branch, the variables $\bar{\eta}_s$, $\bar{Y}_s$ and $[S]_s$ increase to finite maxima at successively larger values of h_s in the interval $0 \leq h_s \leq \hat{\hat{h}}_s$, and then diminish to smaller values at $h_s = \hat{\hat{h}}_s$ where their h_s-derivatives are infinite. The negative branch joins the positive branch smoothly at $h_s = \hat{\hat{h}}_s$ and the values of $\bar{\eta}_s$, $\bar{Y}_s$ and $[S]_s$ decrease to zero as h_s decreases from $\hat{\hat{h}}_s$ to $\hat{h}_s$. These features are illustrated as Figures 6(a), (b) and (e) where we have plotted $\bar{Y}_s$, $\bar{\eta}_s$ and $[S]_s/c_v$ versus h_s for several values of $\theta_o \geq 0$. In these graphs $s_o = 1/16$ and the type 2-curves correspond to values of $\theta_o \leq 30^\circ$; the remaining curves are type 1-curves. In the range $0 < h_s \leq \hat{h}_s$ the magnitude of the transverse magnetic field behind is at most the magnitude of this quantity ahead; in the range $\hat{h}_s \leq h_s \leq \hat{\hat{h}}_s$, $|H_{y,1}|$ is at least equal to $|H_{y,o}|$.

It should be clear from the above that, in contrast to M_f shocks, all state variables behind M_s shocks are bounded. In addition, $\bar{Y}_s$, $V^s_{n,1}/b^s_{n,1}$, $V^s_{n,o}/b_{n,o}$, m_s and $[S]_s$ do not vary directly with $\bar{\eta}_s$ as do $\bar{Y}_f$, $V^f_{n,1}/b^f_{n,1}$, etc. with $\bar{\eta}_f$. In fact, the graphs of $\bar{Y}_s$ and $[S]_s$ versus $\bar{\eta}_s$ are closed loops. The closed **full-lined** curve ($\sin\theta_o = 0.1$) in Figure 4 illustrates this property for

the $\bar{Y}_s$ versus $\bar{\eta}_s$ curves.

It is of interest to note that M_s shocks reduce at $h_s = \hat{h}_s$ (on the minus branch if the shocks are of type 2) to what we shall term <u>180°-intermediate shocks</u> - that is, to intermediate shocks in which the transverse component of the magnetic field behind is oppositely directed to that in front. In the general intermediate shock, as we have seen, the direction of the transverse magnetic field behind is arbitrary.

It should also be observed that the values of $\bar{\eta}_s$, $\bar{Y}_s$ and $[S]_s$ are independent of θ_o in all M_s shocks when $h_s = 2(1-s_o)/\gamma$. In fact, the values of these quantities at this value of h_s are given by the right-hand sides of (2.23).

Since $\bar{\eta}_s$ and h_s are positive quantities, the denominator in the last expression for $V^s_{n,1}/b^s_{n,1}$ in (2.25) is greater than unity. Consequently,

$$(2.35) \qquad V^s_{n,1} < b^s_{n,1}, \qquad 0 < \varepsilon \leq \varepsilon_1;$$

that is, M_s shocks are slow in the sense of (2.4) above. It can also be shown that

$$(2.36) \qquad V^s_{n,o} > c_{s,o}, \qquad 0 < \varepsilon \leq \varepsilon_1,$$

i.e., M_s <u>shocks are super-slow ahead</u>, and that

$$(2.37) \qquad \lim_{\varepsilon \to o} V^s_{n,1} = \lim_{\varepsilon \to o} V^s_{n,o} = c_{s,o},$$

i.e., in the weak shock limit, M_s shocks move with the speed of slow disturbances relative to the flow. The dependence of $V^s_{n,1}$ and $V^s_{n,o}$ on h_s may be described as follows (see Figures 5(c), (d); 6(c), (d)): <u>For sufficiently small values of</u> h_s, $V^s_{n,1}$ <u>is sub-slow</u> ($V^s_{n,1} < c_{s,1}$). At some value of h_s, h^*_s

say,* $V^s_{n,1} = c_{s,1}$; if we limit ourselves to the plus branch we find, for values of h_s in the neighborhood** of h_s^* but greater then h_s^*, that $V^s_{n,1}$ is super-slow ($V^s_{n,1} > c_{s,1}$). $V^s_{n,o}$ increases directly with h_s until $h_s = h_s^*$ where $V^s_{n,o}$ reaches its (unique) maximum value: In $M_s^{(1)}$ shocks $V_{n,o}$ decreases as h_s increases from h_s^* to $\hat{h}_s$; in $M_s^{(2)}$ shocks $V_{n,o}$ decreases as h_s increases from h_s^* to $\hat{\hat{h}}_s$ and continues to decrease (on the negative branch) as h_s decreases from $\hat{\hat{h}}_s$ to $\hat{h}_s$ [see Figures 5(d) and 6(d)]. It is of interest to note that $[S]_s$ reaches a maximum at h_s^*. Thus $V^s_{n,o}$, $m_s = \rho_o V^s_{n,o}$ and $[S]_s$ reach simultaneous maxima at that value of h_s where $V^s_{n,1}$ becomes equal to the slow disturbance speed behind the shock. This property of M_s shocks is illustrated graphically in Figure 7 where we have plotted $V^s_{n,1}/b^s_{n,1}$, $c_{s,1}/b_{n,1}$, $[S]_s/c_v$ against h_s ($s_o = 1/16$, $\theta_o = 30^o$).

There are two kinds of weak M_s shocks, 'proper' and 'improper.' Weak M_s shocks are proper if h_s is small when $\bar{\eta}_s$ is small and h_s vanishes when $\bar{\eta}_s$ vanishes. All other kinds of weak shocks are termed improper. The state behind proper weak M_s shocks may be obtained from that behind weak M_f shocks simply by replacing r_f by $r_s = b^2_{n,o}/c^2_{s,o}$, $c_{f,o}$ by $c_{s,o}$ and h_f by $(-h_s)$ in (2.20) and (2.21). The increase of entropy across proper weak M_s shocks is of the third order in the shock strength.

Improper weak M_s shocks occur in the neighborhood of $h_s = \hat{h}_s \approx 2 \sin \theta_o$. Here,

(2.38) $$\bar{\eta}_s = \frac{\sin \theta_o}{1 - s_o - \gamma \sin^2 \theta_o} \, |h_s - 2 \sin \theta_o| + \ldots, \quad |h_s - 2 \sin \theta_o| \ll 1,$$

---

*It is difficult to obtain a simple algebraic expression for h_s^*. It can be shown, however, that h_s^* is unique and that $h_s^* > 1.5 \sin \theta_o$. The proof of this statement, while not difficult, is long and tedious; we shall therefore omit it entirely.

**Graphical analysis indicates that this is true for all allowable values of h_s which exceed h_s^* on the plus branch and for all values of h_s on the minus branch in $M_s^{(2)}$ shocks [see Figures 5(c) and 6(c)].

if

(2.39) $\qquad s_o \neq 1 - \gamma \sin^2\theta_o ,$

and

(2.40) $\qquad \bar{\eta}_s = \dfrac{(2 \sin \theta_o/\gamma)^{1/2}}{\cos \theta_o} (2 \sin \theta_o - h_s)^{1/2} + \dots, \quad 1 \gg (2 \sin \theta_o - h_s) > 0 ,$

if

(2.41) $\qquad s_o = 1 - \gamma \sin^2\theta_o .$

In both cases

(2.42) $\qquad \bar{Y}_s = \gamma\left[1 + (\gamma-1)\sin^2\theta_o/s_o\right] \bar{\eta}_s + \dots$

and

(2.43) $\qquad [S]_s = \dfrac{c_v \gamma(\gamma-1)\sin^2\theta_o}{s_o} \bar{\eta}_s + \dots .$

Thus, in improper weak M_s shocks the increase of entropy across the shock is of first order in the shock strength parameter $\bar{\eta}_s$.

Finally we note that improper weak M_s shocks reduce to 180^o- intermediate shocks (see p. 21) when $\bar{\eta}_s$ is set equal to zero.

2.6 Limit shocks

2.6.1 Preliminary remarks

Limit shocks are admissible solutions of the hydromagnetic shock relations in their own right. It is most instructive, however, to view them as limits of $M_f(\theta_o)$ and $M_s(\theta_o)$ shocks. According as the various limit shocks serve as limits of $M_f(\theta_o)$ or $M_s(\theta_o)$ shocks we have called them fast or slow. The appropriateness of this nomenclature will become evident as we proceed.

Note that the type relations (2.12), (2.13) and (2.29), (2.30) reduce, in the 0^o-limit, to

(2.44) $\qquad M_f^{(1)}, M_s^{(1)} : \; s_o \geq 1$

$\qquad\qquad M_f^{(2)}, M_s^{(2)} : \; s_o < 1.$

Accordingly, we must expect two 'types' of 0^{o}-limits for M_f shocks and two 'types' of 0^{o}-limits for M_s shocks.

On the other hand, we find, employing the type relations and the fact that s_o is intrinsically positive, that the 90^{o}-limit shocks are all of type 1. Thus we must expect one type of fast 90^{o}-limit shock and one type of slow 90^{o}-limit shock.

2.6.2 Fast 0^{o}-limit shocks

Fast 0^{o}-limit shocks of type 1 ($s_o \geq 1$): Fast pure gas shocks

Consider the curves of Figure 2. As θ_o approaches zero, the vertical line $h_f = \hat{h}_f = 2 \sin \theta_o/(\gamma-1)$, which bounds all curves on the right, moves to the left and coincides in the limit with the vertical axis. Clearly then, if a non-trivial limit shock exists it must correspond to one in which $h_f = 0$. It can be shown that such a non-trivial limit shock does exist and is in fact a gas dynamical* shock (abbreviated G_f shock). In the present notation we find that the state behind a G_f shock is given by the following:

$$G_{f,1} \quad \bar{Y}_f = 2\gamma\, \bar{\eta}_f \,/\, [2 - (\gamma-1)\bar{\eta}_f] \;,$$

(2.45)

$$G_{f,2} \quad m_f^2 = \frac{\mu H_o^2}{\tau_1^f}\left[\frac{2s_o}{2-(\gamma-1)\,\bar{\eta}_f}\right] , \quad \tau_1^f = \frac{1}{\rho_1^f},$$

$$G_{f,3} \quad [u_n]_f = -m\,\tau_1\,\bar{\eta}_f,$$

$$G_{f,4} \quad [u_y]_f = 0,$$

where the range of $\bar{\eta}_f$ is

(2.46) $\qquad 0 < \bar{\eta}_f \leq 2/(\gamma-1).$

Fast gas dynamical shocks are fast in the sense of (2.2); for, using $G_{f,2}$ and the fact that $s_o \geq 1$, we find that

*'Parallel shocks' in the terminology of DeHoffmann and Teller.

$$(2.47) \qquad G_f: \; V^f_{n,1} > b^f_{n,1}, \qquad 0 < \varepsilon \leq \varepsilon_1.$$

Fast 0^o-limit shocks of type 2 ($s_o < 1$): Incomplete switch-on shock - fast gas shock combination. Figures 3 and 4 serve as aids in visualizing the limiting process in M_f shocks of the present type. For the sake of definiteness, consider Figure 3(a): As θ_o approaches zero the upper parts of the $\overline{Y}_f$ versus h_f-curves, which approach the line $h_f = 2 \sin \theta_o / (\gamma-1)$ asymptotically, gradually become steeper and, in the limit, become vertical. The lower parts, on the other hand, approach not the vertical axis but a limit-curve - i.e., the 0^o curve of Figure 3(a). Thus $\overline{Y}_f$ becomes independent of h_f on the upper parts of the $\overline{Y}_f$ versus h_f-curves but retains its dependence on h_f on the lower parts. Similar remarks apply to the $\overline{\eta}_f$, $V_{n,1}/b_{n,1}$, etc. versus h_f-curves of Figure 3.

It can be shown that the limit shock is composed of two 'incomplete' parts: the incomplete switch-on shock (abbreviated Sw shock) and the incomplete fast gas shock. The state behind a G_f shock is determined by (2.45). Moreover, using (2.45) $G_{f,2}$, it is easy to show that the gas shock is fast - i.e.,

$$(2.48) \qquad V_{n,1} \geq b_{n,1},$$

if, and only if, $\overline{\eta}_f$ is restricted to the range*

$$(2.49) \qquad \frac{2}{\gamma-1}(1-s_o) = \overline{\eta}_{crit} \leq \overline{\eta}_f < 2/(\gamma-1), \qquad s_o < 1,$$

or, alternatively, if, and only if, $\overline{Y}_f$ is restricted to the range*

$$(2.50) \qquad \frac{2\gamma(1-s_o)}{(\gamma-1)s_o} \equiv \overline{Y}_{crit} \leq \overline{Y}_f < \infty, \qquad s_o < 1.$$

The incomplete G_f-shock is 'completed' by the incomplete Sw shock.

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*It is because of this limitation on the ε-range that a fast pure gas shock is called 'incomplete' when $s_o < 1$. Sw shocks are termed incomplete for similar reasons.

The state behind an incomplete Sw shock is determined by:*

$$\text{Sw}_1 \quad h_f^2 = 2\bar{\eta}_f \left\{ (1-s_o) - (\gamma-1)\,\bar{\eta}_f/2 \right\}, \qquad h_f > 0,$$

$$\text{Sw}_2 \quad \bar{Y}_f = \gamma\bar{\eta}_f \left\{ 1 + (\gamma-1)\bar{\eta}_f/2s_o \right\},$$

$$(2.51) \qquad \text{Sw}_3 \quad [u_n]_f = -b_{n,o} \left\{ \eta_f^{1/2} - \eta_f^{-1/2} \right\},$$

$$\text{Sw}_4 \quad [u_y]_f = b_{n,1}^f\, h_f = b_{n,o}\, \eta_f^{-1/2} h_f,$$

$$\text{Sw}_5 \quad m_f^2 = \rho_1^f\, \mu H_{n,o}^2 .$$

The Sw shock, considered by itself, is an intermediate shock; for, using Sw_5, we find:

$$(2.52) \qquad V_{n,1} = b_{n,1}.$$

The range of validity of (2.51) and (2.52) is:

$$(2.53) \qquad 0 < \bar{\eta}_f \leq \bar{\eta}_{crit} = 2(1-s_o)/(\gamma-1), \qquad s_o < 1,$$

or alternatively

$$(2.54) \qquad 0 < \bar{Y}_f \leq \bar{Y}_{crit} \leq 2\gamma(1-s_o)/(\gamma-1)s_o, \qquad s_o < 1.$$

It is easily verified that the state behind the Sw shock passes continuously into the state behind the incomplete G_f shock as the shock strength parameter passes through the critical value - e.g., as $\bar{\eta}_f$ passes through

---

*Essentially these relations were derived by Friedrichs directly from the basic equations; they appeared in unpublished notes which he kindly placed at our disposal. Some properties of these solutions were also mentioned by Friedrichs in [2]. A more detailed discussion may be found in [8] where it is shown that Sw shocks play an important role in the resolution of an initial shear flow discontinuity.

$\bar{\eta}_{crit} = 2(1-s_o)/(\gamma-1)$. In the complete range $0 < \bar{\eta}_f \leq 2/\gamma-1$, the ratio $\bar{Y}_f$ increases monotonically with $\bar{\eta}_f$. This is illustrated in Figure 4 (see curve ODBC∞) where we have plotted $\bar{Y}_f$ against $\bar{\eta}_f(s_o = 1/16)$; ODB is the Sw part, BC∞ is the G_f part of ODBC∞. Regarded as functions of h_f, the variables $\bar{\eta}_f$, $\bar{Y}_f$, m_f, $[S]_f$, on the Sw branch, are double-valued. [see Figure 3.]

The relationships of $V^f_{n,1}$ and $V^f_{n,o}$ behind and ahead of the switch-on shock—fast gas shock (abbreviated SwG_f) combination to the small disturbance speeds ahead of and behind the shock are given by the following:

$$(2.55) \qquad SwG_f: \quad \begin{aligned} &b^f_{n,1} = V^f_{n,1} < c_{f,1}, \quad 0 < \bar{\eta}_f \leq \bar{\eta}_{crit}, \\ &b^f_{n,1} \leq V^f_{n,1} < a_1, \quad \bar{\eta}_{crit} \leq \bar{\eta}_f \leq \frac{2}{\gamma-1}, \end{aligned}$$

$$(2.56) \qquad SwG_f: \quad V^f_{n,o} > b_{n,o} > a_o, \quad 0 < \varepsilon \leq \varepsilon_1.$$

The state behind weak switch-on shocks is determined by:

$$\begin{aligned} &\Delta Sw_1 \qquad h_f^2 = 2(1-s_o)\,\bar{\eta}_f + \dots, \\ &\Delta Sw_2 \qquad \bar{Y}_f = \gamma\,\bar{\eta}_f + \dots, \\ (2.57)\quad &\Delta Sw_3 \qquad [u_n]_f = -b^f_{n,o}\,\bar{\eta}_f + \dots, \\ &\Delta Sw_4 \qquad [u_y]_f = b^f_{n,o} h_f + \dots, \\ &\Delta Sw_5 \qquad V^f_{n,1} = V^f_{n,o} = b_{n,o} + \dots . \end{aligned}$$

In addition, we find:

$$(2.58) \qquad [S]_f = \frac{c_v\,\gamma(\gamma-1)(1-s_o)}{2s_o}\,\bar{\eta}_f^2 + \dots = \frac{c_v\,\gamma(\gamma-1)h_f^4}{8s_o(1-s_o)} + \dots .$$

Thus, the rise in entropy across an Sw shock is of the second order in the excess

density or pressure ratio and of the fourth order in the variable h_f.

2.6.3 Slow 0^o-limit shocks

Slow 0^o-limit of type 1 ($s_o \geq 1$): Continuous transition

In the $\theta_o = 0^o$ limit all discontinuities of the $M_s^{(1)}$ shock vanish, producing a continuous transition across the surface $\mathcal{S}(t)$. The limiting process is most easily visualized with the aid of curves of Figure 5. In Figure 5(a), for example, the ranges of $\bar{Y}_s$ and of h_s decrease with decreasing θ_o and reduce in the 0^o-limit to zero.

Slow 0^o-limit shocks of type 2 ($s_o < 1$): Slow gas shock—switch-on shock combination.

As was the case with $M_f^{(2)}$ shocks, slow 0^o-limit shocks of type 2 consist of two incomplete parts. One part is an incomplete slow gas shock (abbreviated G_s shock); the other part is an incomplete switch-on shock.

The state behind the incomplete G_s shock is determined by (2.45) with 'f' replaced by 's'. It is slow because $\bar{\eta}_s$ is restricted to the range

$$(2.59) \qquad 0 < \bar{\eta}_s \leq \bar{\eta}_{crit} = 2(1-s_o)/(\gamma-1),$$

[cf. (2.53)] and because in this range (and in this range only)

$$(2.60) \qquad v^s_{n,1} \leq b^s_{n,1} ,$$

the equality holding only when $\bar{\eta}_s = \bar{\eta}_{crit}$.

The state behind the Sw shock is determined by (2.51)-(2.54) with h_f replaced by h_s. This Sw shock differs from the one in the SwG_f combination in one respect only; here, $H_{y,1}$ is negative whereas $H_{y,1}$ exceeds zero in the

SwG_f combination.*

In the $(\bar{\eta}_s, \bar{Y}_s)$-plane the graph of $\bar{Y}_s$ against $\bar{\eta}_s$ is a closed loop [see Figure 4]. This loop consists of the arc OAB (G_s shock) and the arc BDO (Sw shock). The smooth, closed, nearby loop is an $M_s^{(2)}$ curve corresponding to a small value of θ_o, namely, $\theta_o = \arcsin(0.1)$.

The limiting process is graphically portrayed in Figure 6. Consider Figure 6(a) for example: These curves may be divided into two arcs, one in the range $0 < h_s \leq \tilde{h}_s(\theta_o)$, the other in the range $\tilde{h}(\theta_o) \leq h_s \leq h_s$, where $\tilde{h}_s(\theta_o)$ is the value of h_s for which $\bar{Y}_s$ is maximum, say. As θ_s approaches zero the first arc becomes steeper and in the limit coincides with part of the $\bar{Y}_s$-axis. The remaining part reduces to the switch-on shock.

The relationship of $V^s_{n,1}$ and $V^s_{n,o}$ behind and ahead of the slow gas shock-switch-on shock combination (abbreviated G_sSw combination) is given by the following:

$$(2.61)\quad G_sSw:\ V^s_{n,1} \leq \min(a^s_1, b^s_{n,1}),\quad 0 < \bar{\eta} \leq \bar{\eta}_{crit}\quad (\text{on } G_s \text{ branch}),$$

$$V^s_{n,1} = b^s_{n,1},\quad 0 < \bar{\eta}_s \leq \bar{\eta}_{crit}\ (\text{on Sw branch}),$$

$$(2.62)\quad G_sSw:\ V^s_{n,o} > a_o,\qquad 0 < \bar{\eta}_s \leq \bar{\eta}_{crit}\ ,\quad (\text{on } G_s \text{ and Sw branches})\ .$$

$V^s_{n,o}$ is at most $b_{n,o}$ when

---

*The basic shock relations do not uniquely determine the direction of the transverse magnetic field. If, however, Sw shocks are regarded as 0^o-limits of $M_f(\theta_o)$ and $M_s(\theta_o)$ shocks, then it follows (since we are always assuming $\theta_o > 0$) that $H_{y,1} > 0$ in fast Sw shocks and $H_{y,1} < 0$ in slow Sw shocks.

(2.63) $\quad 0 < \bar{\eta}_s \leq 2(1-s_o)/(2s_o + \gamma - 1)\,, \qquad$ (on G_s branch),

and at least $b_{n,o}$ in the ranges

(2.64)
$$2(1-s_o)/(2s_o + \gamma - 1) \leq \bar{\eta}_s \leq \bar{\eta}_{crit}\,, \qquad \text{(on } G_s \text{ branch)}\,,$$
$$0 < \bar{\eta}_s \leq \bar{\eta}_{crit}\,, \qquad \text{(on Sw branch)}\,.$$

2.6.4 Fast 90^o-limit shocks (perpendicular shocks)

Fast 90^o-limit shocks are, in the terminology of de Hoffmann and Teller, the so-called perpendicular shocks. We have seen that 90^o-limit shocks are of type 1. The state behind fast, perpendicular shocks is determined by the following relations:

(2.65)

$$P_{f,1} \quad \bar{\eta}_f = h_f\,,$$

$$P_{f,2} \quad \bar{Y}_f = \frac{\gamma\, \bar{\eta}_f\left[1 + (\gamma-1)\, \bar{\eta}_f^2/4s_o\right]}{\left[1 - (\gamma-1)\, \bar{\eta}_f/2\right]}\,, \qquad 0 < \bar{\eta}_f \leq 2/(\gamma-1)\,,$$

$$P_{f,3} \quad V^f_{n,1} = \frac{V^f_{n,o}}{\eta_f} = b_o \frac{\left[1+s_o + (2-\gamma)\, \bar{\eta}_f/2\right]^{1/2}}{\eta_f^{1/2}\left[1-(\gamma-1)\bar{\eta}_f/2\right]^{1/2}}\,, \qquad b_o = \sqrt{\mu H_o^2 \rho_o^{-1}}\,,$$

$$P_{f,4} \quad [u_n]_f = (V^f_{n,1} - V^f_{n,o}) = -b_o \frac{\left[1+s_o+(2-\gamma)\, \bar{\eta}_f/2\right]^{1/2}}{\left[1-(\gamma-1)\, \bar{\eta}_f/2\right]^{1/2}} (\eta_f^{1/2} - \eta_f^{-1/2})\,,$$

$$P_{f,5} \quad [u_y]_f = 0.$$

It follows from $P_{f,3}$ that $V^f_{n,o}$ varies in the same sense as $\bar{\eta}_f$. From this we conclude that

(2.66) $$V^f_{n,o} > b_{n,o}(1+s_o)^{1/2} = (b^2_{n,o} + a^2_o)^{1/2} \equiv c_{f,o}.$$

Comparing the expression for $c_{f,1}$ - i.e.,

$$c_{f,1} = \frac{b_{n,o}}{\eta^{1/2}} \left\{ \frac{1+s_o\left[1+(\gamma+1)\bar{\eta}_f/2\right] + \bar{\eta}_f(3-\gamma)/2 + \bar{\eta}_f^{\,2}(2-\gamma)\left[1-(\gamma-1)\bar{\eta}_f/4\right]}{1-(\gamma-1)\bar{\eta}_f/2} \right\}^{1/2}$$

with the expression for $V^f_{n,1}$ in $P_{f,3}$ we find, in addition

$$(2.67) \qquad V^f_{n,1} < c_{f,1}, \qquad\qquad 0 < \varepsilon \leq \varepsilon_1.$$

Weak fast perpendicular shock equations may be derived from (2.20)-(2.22) simply by setting $\theta_o = 90^\circ$ and $r_f = b^2_{n,o}/c^2_{f,o} = 0$ in these equations.

2.6.5 Slow 90°-limit (contact discontinuity)

The 90°-limit of an $M_s^{(1)}$ shocks is a special case of the contact discontinuity solution characterized in (2.6). As in (2.6) we have

$$(2.68) \qquad m^2_s = 0, \ [u_n]_s = 0, \ [p + \mu H^2_{tr}/2]_s = 0, \quad [\vec{u}_{tr}]_s \text{ arbitrary.}$$

In contrast to (2.6), however, $[\tau]_s$ and $[S]_s$ are no longer arbitrary. For example, it can be shown that

$$(2.69) \qquad \bar{\eta}_s = \frac{(2-h_s)h_s}{2s_o + (\gamma-1)h_s}.$$

Since $[S]_s/c_v = \log(Y_s/\eta^\gamma_s)$, $[S]_s$ also depends on h_s.

2.7 Concluding remarks - Survey of the literature

2.7.1 Concluding remarks

We add here several remarks with the object of completing the qualitative picture of hydromagnetic shocks. The first is that the excess density ratio in all compressive shocks never exceeds the gas dynamical upper limit $2/(\gamma-1)$. This upper limit is actually assumed in M_f shocks; in M_s shocks

the maximum value of the excess density ratio is less than $2/(\gamma-1)$. The second remark may be expressed as follows: The pressure behind a magnetic shock corresponding to a given value of the excess density ratio is, owing to the presence of transverse magnetic fields, greater than the pressure behind a pure gas shock at that value of the excess density ratio [see Figure 4]. This result is an immediate consequence of the relation [cf. Lüst [4]a]

$$(2.70) \qquad \bar{Y} = \frac{2\gamma\,\bar{\eta}}{2-(\gamma-1)\,\bar{\eta}} + \frac{\gamma(\gamma-1)h^2\,\bar{\eta}}{2s_o\left[2-(\gamma-1)\,\bar{\eta}\,\right]}\,, \qquad 0 < \bar{\eta} \le 2/(\gamma-1)\,,$$

which is the hydromagnetic analog of the gas dynamical Hugoniot relation. The third remark involves a result already implicit in the above development but which nevertheless deserves to be singled out for special emphasis. It is: the magnetic field behind compressive shocks is finite for all values of the shock strength parameter. Finally we remark that the entropy behind a compressive shock varies in the same sense as the mass flux or, alternatively, as the normal flow velocity (relative to the shock) in the region ahead of the shock. This result follows directly from the differential relation [see Subsection 3.2]

$$(2.71) \qquad T_1 d[S] = \left\{ [\tau]^2 + [\tau H_y]^2/H_n^2 \right\} \text{ mdm},$$

which we believe has not appeared in the literature thus far. It is valid for all compressive shocks; the last term is absent in pure gas shocks and in perpendicular shocks.

2.7.2 Survey of the literature

The study of compressive hydromagnetic shocks of finite amplitude was initiated by de Hoffmann and Teller[1], who as already mentioned, derived the hydromagnetic analogs (relativistic and non-relativistic) of the Rankine-Hugoniot relations of gas dynamics. Shock wave solutions of these relations, corresponding

to special orientations of the magnetic fields in front of and behind the shock front - explicitly, parallel, perpendicular and symmetrical shocks (i.e., our intermediate shocks)-were discussed in some detail by these authors and the extreme relativistic and non-relativistic behavior was examined. The steady shock wave solutions of the non-relativistic set of hydromagnetic discontinuity relations was then taken up by Helfer[3] who discussed shock wave solutions corresponding to more general orientations of the magnetic field (i.e., oblique shocks). Helfer's analysis involves, in addition to $Y = p_1/p_0$, the three parameters Q, s and x, which in our notation and system of units [see Subsection 1.2] may be expressed as follows:

$$Q = \frac{\Upsilon}{2s_0}$$

$$(2.72) \qquad s = \tan\theta_0 \;, \qquad 0 < \theta_0 < 90^\circ ,$$

$$x = \tan\theta_1/\tan\theta_0 \;.$$

Helfer divides magnetic shocks into three classes a) $x > 1$, b) $0 < x \leq 1$, c) $x \leq 0$, and derives bounds on s and x which serve to delimit the ranges of s and x in which the shocks are admissible. Helfer's results for the case $\gamma = 5/3$ are summarized graphically in Figures 3-7 of his paper where he plots $Q = Q(\overline{Y}, x, s)$ versus $\overline{Y}$ for fixed s and for several values of x. Using Helfer's results one could obtain a plot of $\overline{Y}$ versus x for several values of s and fixed Q. It would then perhaps be possible to compare our results with his. Such a procedure would, however, take us too far afield. We shall, instead, be satisfied with noting the following relations between Helfer's magnetic shocks and ours:

$$
\text{(2.73)}\quad
\begin{array}{ll}
\text{a) } x > 1 & \begin{cases} M_f^{(1)}, & 0 < h_f \leq \hat{h}_f = 2 \sin \theta_o/(\gamma-1), \\ M_f^{(2)}, & 0 < h_f \leq \hat{\hat{h}}_f, \ [\text{cf. } (2.15)]; \end{cases} \\
\text{b) } 0 < x \leq 1 & \begin{cases} M_s^{(1)}, & 0 < h_s \leq \sin \theta_o , \\ M_s^{(2)}, & 0 < h_s \leq \sin \theta_o ; \end{cases} \\
\text{c) } x \leq 0 & M_s^{(2)}, \quad \sin \theta_o < h_s \leq \hat{\hat{h}}_s, \ [\text{cf. } (2.33)] .
\end{array}
$$

The work of Lüst [4(b)] is mainly numerical in character. Lüst's analysis is based on the fact that the magnetic shock velocity and all quantities behind may be calculated, for a given state in front, once a suitably defined Mach number is prescribed and once the dependence of the density ratio on this Mach number is known. Lüst shows that the density ratio satisfies a cubic equation whose coefficients are cubic polynomials in the square of the Mach number. He then solves this equation numerically for the case of $\gamma = 5/3$ and determines the velocity of the shock and the state behind the shock. Because our choices of shock strength parameters differ from those Lüst has employed (i.e. U_n/c_f, U_n/c_s, U_n/b_n), a direct comparison of numerical results is again not possible. Nevertheless, by making use of the relations $V^f_{n,o} > c_{f,o}$ (in M_f shocks), $V_{n,o} = b_{n,o}$ (in intermediate shocks), $V^s_{n,o} > c_{s,o}$ (in M_s shocks) [cf. (2.18), (2.9), (2.36)] it is possible to identify our M_f, Alfvén and M_s shocks with the three types of shocks distinguished by Lüst [see [4(b)] p. 134], namely, those in which $\lim_{\varepsilon \to o} V_{n,o} = c_{f,o}$, those in which $V_{n,o} = b_{n,o}$ and those in which $\lim_{\varepsilon \to o} V_{n,o} = c_{s,o}$. This identification having been made, it appears that there is complete agreement on a qualitative level between Lüst's numerical results and our analytical and numerical results.

The inspiration for the present work comes from the report of Friedrichs' [2].

Friedrichs (as does Lüst) finds three types of shocks, the so-called fast, intermediate and slow, which, in the weak shock limit, move with the fast, intermediate and slow disturbance speeds respectively. In contrast to Lüst's investigation, however, Friedrichs' work enjoys the advantage of being chiefly analytical in character. Our M_f shocks correspond to Friedrichs' fast shocks. This is to be expected since the essential features of his fast shocks, namely $|H_{y,1}| > |H_{y,o}|$, $V_{n,1} > b_{n,1}$, have been built into our definition of M_f shocks. Our M_s shocks are equivalent to Friedrichs' slow shocks provided h_s is restricted to the range,

$$0 < h_s \leq \sin \theta_o \ .$$

It is important to note, however, that Friedrichs limits his slow shock entirely to this range and finds as a consequence,

$$V^s_{n,1} \leq c_{s,1}.$$

This limitation of the range of h_s is unnecessarily restrictive; for, as we have seen, the appropriate range is $0 < h_s \leq 2 \sin \theta_o$ in $M_s^{(1)}$ shocks and $0 < h_s \leq \hat{\hat{h}}_s$ in $M_s^{(2)}$ shocks where $\hat{\hat{h}}_s > 2 \sin \theta_o$ [see Subsection 2.5]. While it is true that $V^s_{n,1} \leq c_{s,1}$ in the range $0 < h_s \leq \sin \theta_o$, it is not true throughout the complete admissible range at h_s. Incidentally, this resolves a discrepancy between a result of Lüst and one due to Friedrichs. Lüst shows, by numerical means, that slow waves exist in the region $b_{n,1} \geq V^s_{n,1} > c_{s,1}$ and, more important still, that the slow wave is connected to the 180°-intermediate shock [see Subsection 2.5]. The fact that 180°-intermediate shocks are not so connected in Friedrichs' analysis is due to his unnecessary limitation of the h_s- range.

Our analysis differs from Friedrichs' in several other respects. First,

it should be observed that Friedrichs specifies the average state (that is, the average of like quantities on both sides of the shock); the shock velocity and the state behind are then determined in terms of the average state and a suitable shock strength parameter. This type of solution of the shock relations has the advantage of simplifying the analysis and of leading directly to many important properties of the various shocks. In initial value problems, however, it is the state in front that is usually known; shock wave solutions in which the state in front is specified are therefore preferable in this connection.

Basically, however, the advantage of our procedure stems from the fact that, given the state in front, we have been able to determine the state behind magnetic shocks in terms of relatively simple algebraic functions of the jump in the magnetic field. By varying the state in front we are thus able to obtain a comparatively complete picture of the family of all shock wave solution of the basic discontinuity relations. In addition, important properties of these solutions may readily be derived. Thus, for example, as we have seen, the magnetic field behind magnetic shocks is finite for all values of the shock strength parameter; an analytic proof of this fact seems never to have been given before. Our particular representation also enables us to obtain a more complete classification of the various shock wave solutions. For instance, our classification of magnetic shocks according to 'type' (see Sections 2.4 and 2.5) may be regarded as a refinement of the scheme of classification given by Friedrichs. As we have seen, this refinement has the virtue of clarifying the connection of the slow and fast magnetic shocks with their zero-degree limits.

Finally, it should be mentioned that our definitions of fast and slow shocks differ somewhat from the corresponding definitions given by Friedrichs.

We define a shock to be fast if $u_{n,1} - U_n \geq b_{n,1}$, slow if $u_{n,1} - U_n \leq b_{n,1}$, and, in addition, we require the inequality to hold in at least a proper subrange of the shock strength parameter range; Friedrichs does not include this last requirement. Our definitions allow us to regard the 0^o-limits (see Sections 2.6.2 and 2.6.3) of slow and fast shocks as slow and fast, respectively. These definitions and the 'type' relations enable us to place 0^o-limit shocks in their proper context within the family of solutions obtained by varying the direction of the magnetic field in front.

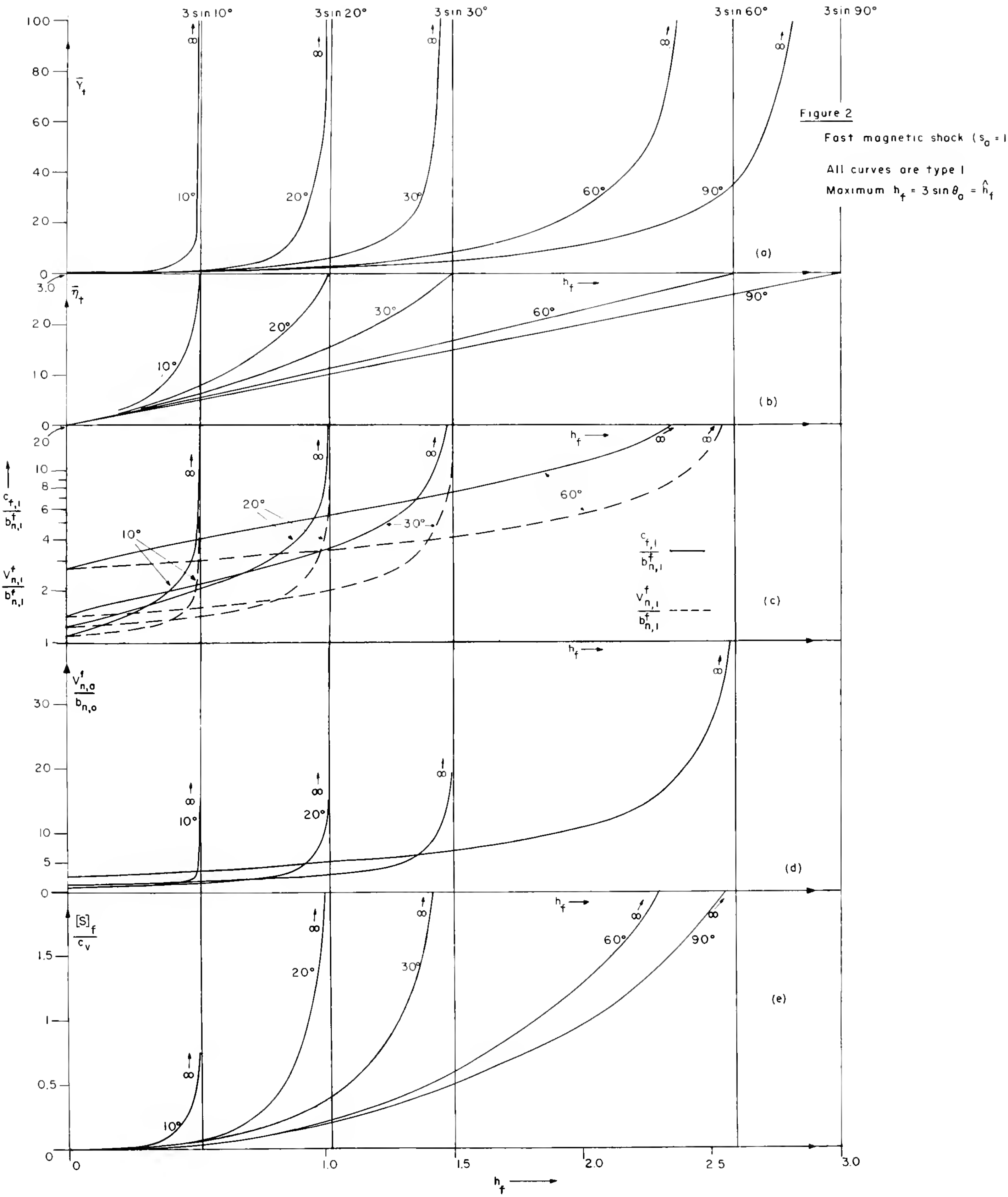
3 sin 10°
3 sin 20°
3 sin 30°
3 sin 60°
3 sin 90°
Figure 2
Fast magnetic shock ($s_a = 1$)
All curves are type I
Maximum $h_f = 3 \sin\theta_a = \hat{h}_f$
$\bar{Y}_t$
$\bar{\eta}_t$
$\frac{c_{f,1}}{b^f_{n,1}}$
$\frac{V^f_{n,1}}{b^f_{n,1}}$
$\frac{V^f_{n,a}}{b_{n,a}}$
$\frac{[S]_f}{c_v}$
(a)
(b)
(c)
(d)
(e)
h_f
10°
20°
30°
60°
90°
∞
0
0.5
1.0
1.5
2.0
2.5
3.0

Figure 2. Illustrating the dependence in M_f shocks of $\overline{Y}_f$, $\overline{\eta}_f$, $V^f_{n,1}/b^f_{n,1}$ $c_{f,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, $[S]_f/c_v$ on the shock strength parameter h_f for several values of the parameter θ_o. Here $s_o = 1$ and θ_o varies from 10^o- 90^o. All curves are of type 1. The maximum value of h_f is $\hat{h}_f \equiv 2 \sin \theta_o/\gamma-1 = 3 \sin \theta_o$. As h_f increases to $\hat{h}_f$, $\overline{\eta}_f$ increases monotonically to $2/\gamma-1 = 3$; the remaining dependent variables increase monotonically to infinity, approaching the line $h_f = \hat{h}_f$ asymptotically. As θ_o approaches zero the asymptotes and their associated curves approach the vertical axis. The 0^o-limit curves are independent of h_f and correspond to the fast pure gas shock (see Section 2.6.2).

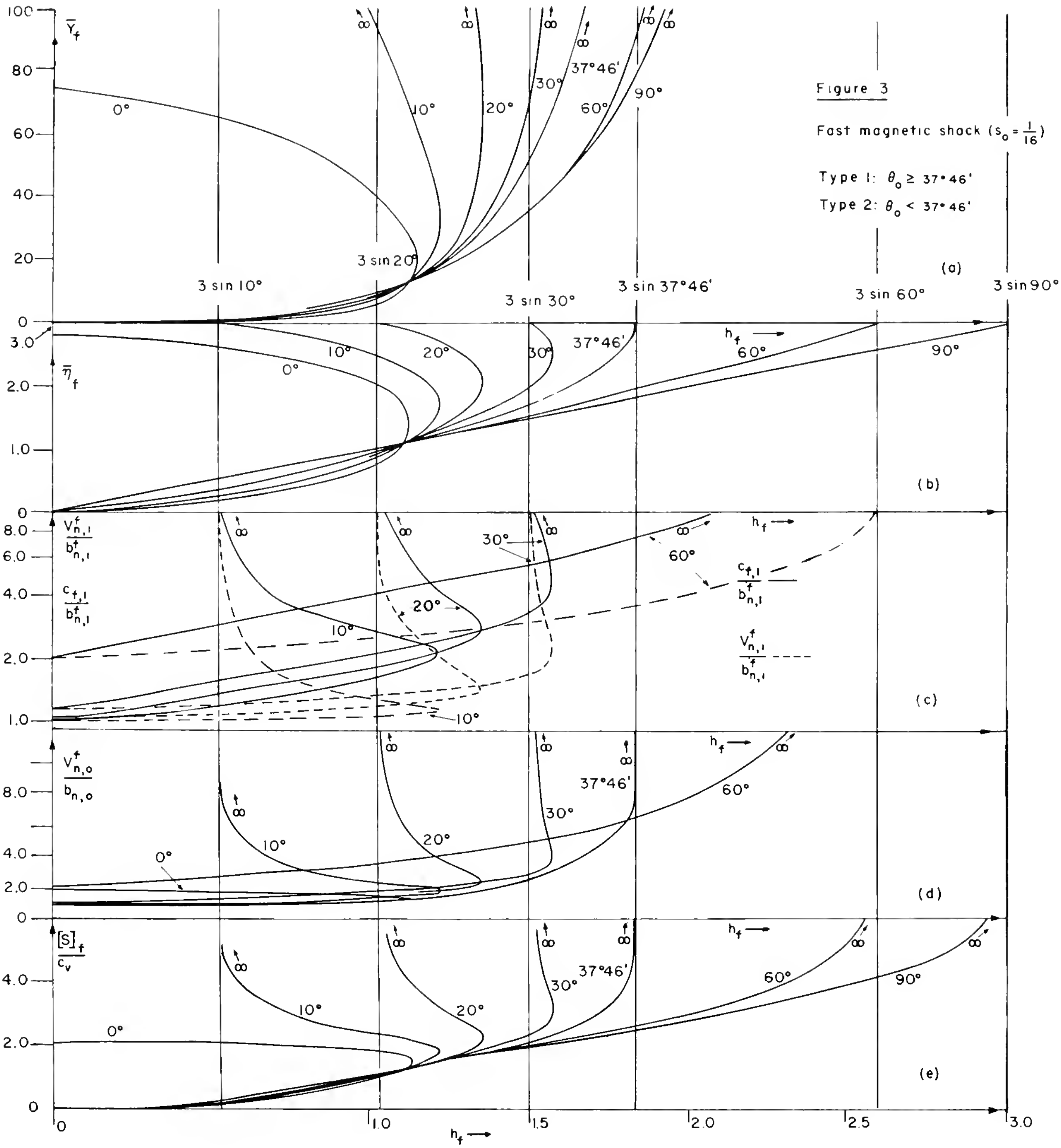

Figure 3
Fast magnetic shock ($s_o = \frac{1}{16}$)
Type 1: $\theta_o \geq 37°46'$
Type 2: $\theta_o < 37°46'$
(a)
(b)
(c)
(d)
(e)
3 sin 10°
3 sin 20°
3 sin 30°
3 sin 37°46'
3 sin 60°
3 sin 90°
$\bar{Y}_f$
$\bar{\eta}_f$
$\frac{V^f_{n,1}}{b^f_{n,1}}$
$\frac{c_{f,1}}{b^f_{n,1}}$
$\frac{V^f_{n,o}}{b_{n,o}}$
$\frac{[S]_f}{c_v}$
h_f
0°
10°
20°
30°
37°46'
60°
90°

<u>Figure 3</u>. Illustrating the dependence in M_f shocks of $\bar{Y}_f$, $\bar{\eta}_f$, $V^f_{n,1}/b^f_{n,1}$, $c_{f,1}/b^f_{n,1}$, $V^f_{n,o}/b_{n,o}$, $[S]_{f/c_v}$ on the shock strength parameter h_f for several values of θ_o. Here, $s_o = 1/16$; curves corresponding to values of $\theta_o < 37^o46'$ are type 2 curves; $\theta_o \geq 37^o46'$, type 1 curves. The 0^o-curves represent the state behind an incomplete switch-on shock. The maximum value of h_f in $M_f^{(1)}$ shocks is $\hat{h}_f = 2 \sin \theta_o/(\gamma-1) = 3 \sin \theta_o$; in $M_f^{(2)}$ shocks, the maximum value is $\hat{\hat{h}}_f > \hat{h}_f$ [see (2.15)]. As h_f approaches h_f and $\bar{\eta}_f$ approaches $2/(\gamma-1) = 3$, the remaining dependent variable becomes infinite, approaching $h_f = \hat{h}_f$ asymptotically. As θ_o approaches zero, the upper parts of all curves become steeper and in the limit become vertical. The lower parts, on the other hand, approach not the vertical axis but the incomplete switch-on shock. The limiting solution is the incomplete gas shock—switch-on shock combination (see Section 2.6.2 and Figure 4).

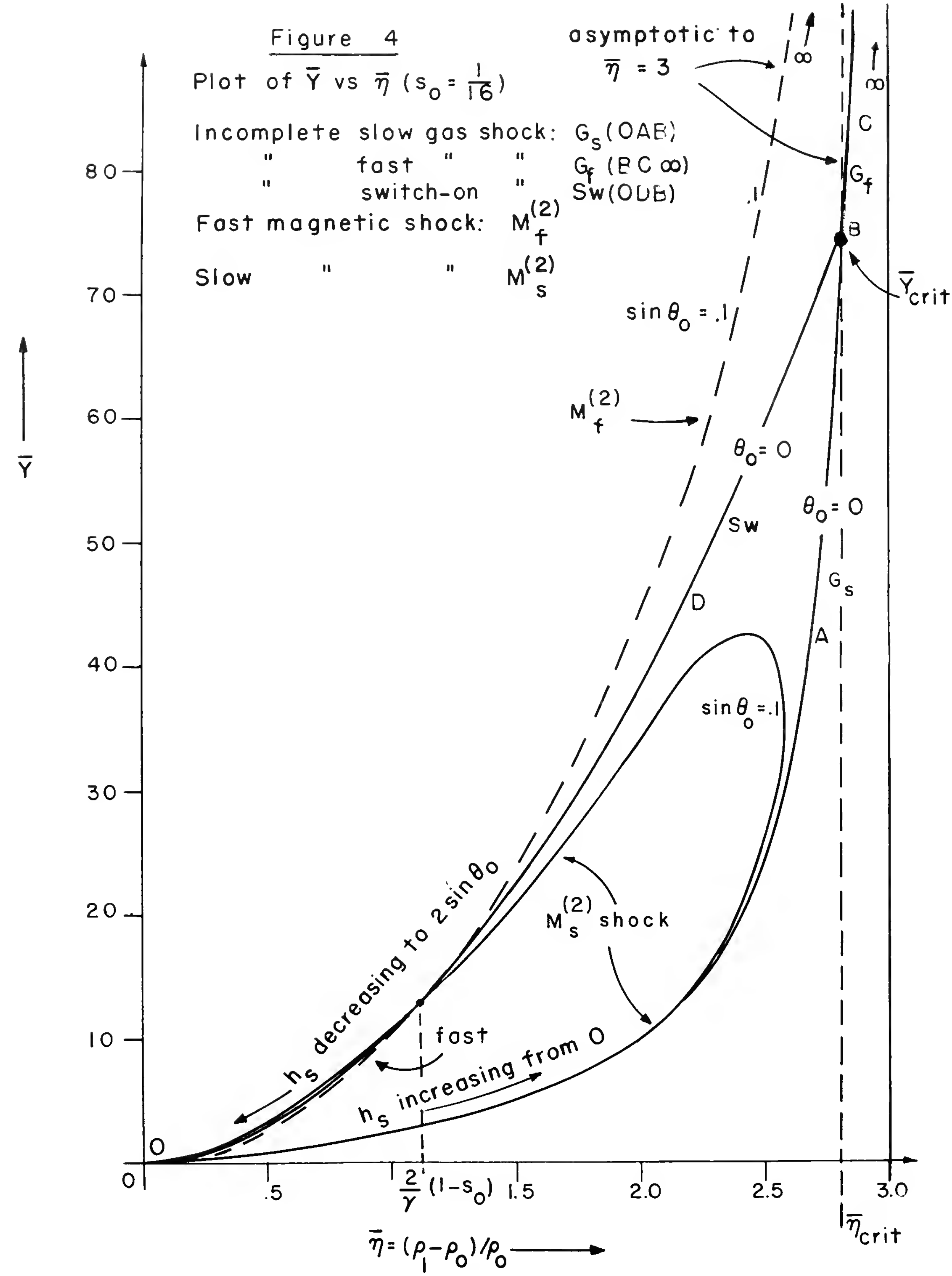
Figure 4
Plot of $\overline{Y}$ vs $\overline{\eta}$ ($s_o = \frac{1}{16}$)
Incomplete slow gas shock: G_s(OAB)
" fast " " G_f (BC ∞)
" switch-on " Sw(ODB)
Fast magnetic shock: $M_f^{(2)}$
Slow " " $M_s^{(2)}$
asymptotic to
$\overline{\eta}$ = 3
$\overline{Y}_{crit}$
$\sin\theta_o = .1$
$\theta_o = 0$
$M_s^{(2)}$ shock
h_s decreasing to $2\sin\theta_o$
h_s increasing from 0
fast
$\frac{2}{\gamma}(1-s_o)$
$\overline{\eta}_{crit}$
$\overline{\eta} = (\rho_1 - \rho_o)/\rho_o$
$\overline{Y}$

<u>Figure 4</u>. Illustrating the relationship between $\overline{Y}$ and $\overline{\eta}$ in SwG_f and G_sSw shocks and in neighboring $M_f^{(2)}$ and $M_s^{(2)}$ shocks. The curve $OABC\infty$ is a pure gas shock; the arc OAB is the incomplete G_s shock; the remaining arc $BC\infty$ is the incomplete G_f shock. The arc ODB represents the incomplete switch-on shock; the closed loop OABDO represents the G_sSw combination; the curve $ODBC\infty$ the SwG_f combination. The $M_s^{(2)}$ and $M_f^{(2)}$ shocks reduce in the 0°-limit to G_sSw and SwG_f shocks, respectively.

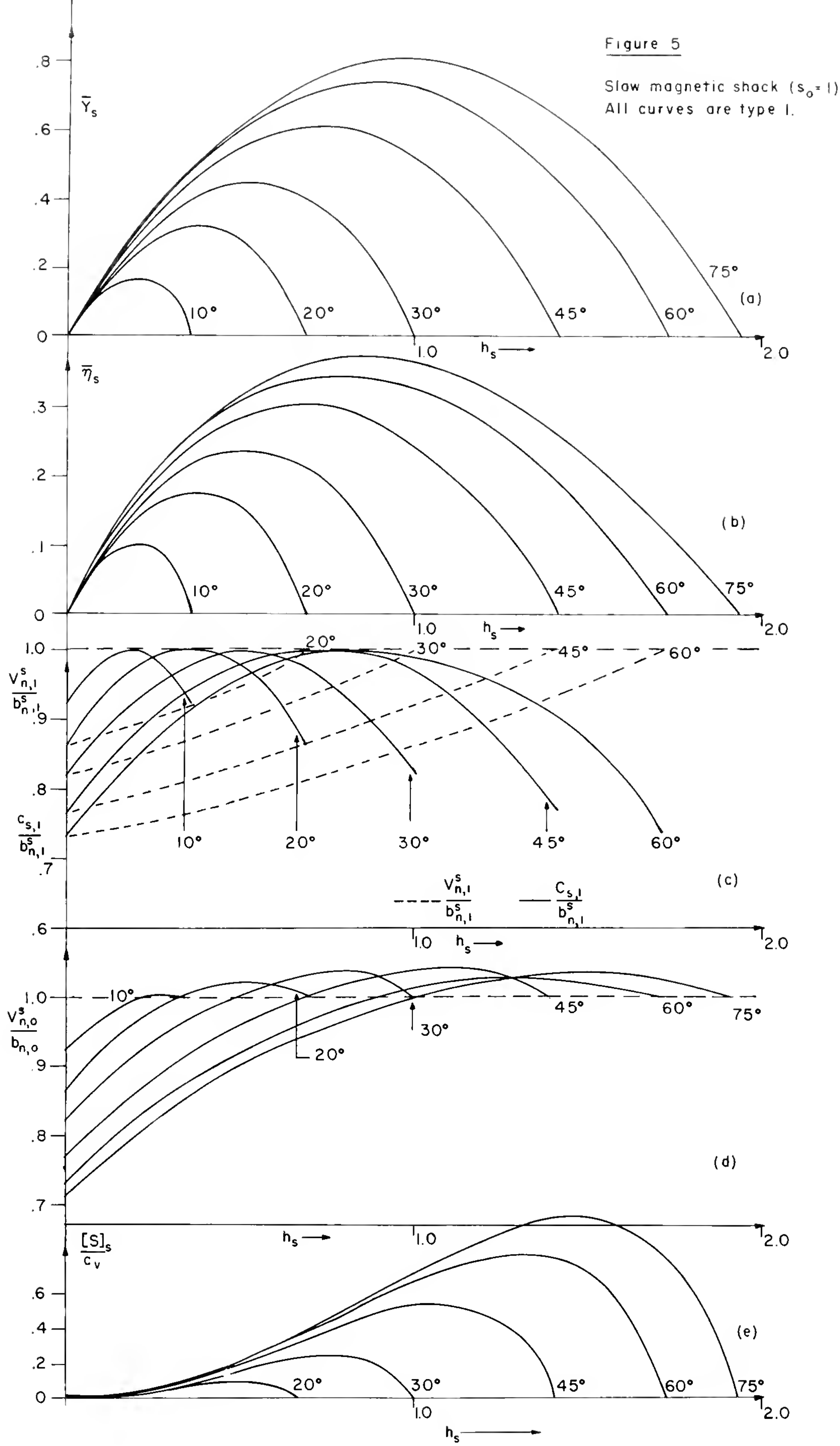
Figure 5
Slow magnetic shock ($s_o = 1$)
All curves are type I.
$\bar{Y}_s$
.8
.6
.4
.2
0
10°
20°
30°
45°
60°
75°
(a)
1.0
h_s
2.0
$\bar{\eta}_s$
.3
.2
.1
(b)
$\frac{V^s_{n,1}}{b^s_{n,1}}$
$\frac{c_{s,1}}{b^s_{n,1}}$
1.0
.9
.7
.6
(c)
$\frac{V^s_{n,o}}{b_{n,o}}$
(d)
$\frac{[S]_s}{c_v}$
(e)

Figure 5. Illustrating the dependence in M_s shocks of $\bar{Y}_s$, $\bar{\eta}_s$, $V^s_{n,1}/b^s_{n,1}$, $c_{s,1}/b^s_{n,1}$, $V^s_{n,o}/b_{n,o}$, $[S]_s/c_V$ on the shock strength parameter h_s for several values of the parameter Θ_o. Here, $s_o = 1$; all curves are of type 1. The maximum value of h_s in $M_s^{(1)}$ shocks is $\hat{h}_s = 2 \sin \Theta_o$. As h_s approaches $\hat{h}_s$ the $M_s^{(1)}$ shock reduces to a 180°-intermediate shock - i.e., to an intermediate shock in which the transverse magnetic field behind is directed oppositely to that ahead. In the 0° limit, all discontinuities of the $M_s^{(1)}$ shock vanish.

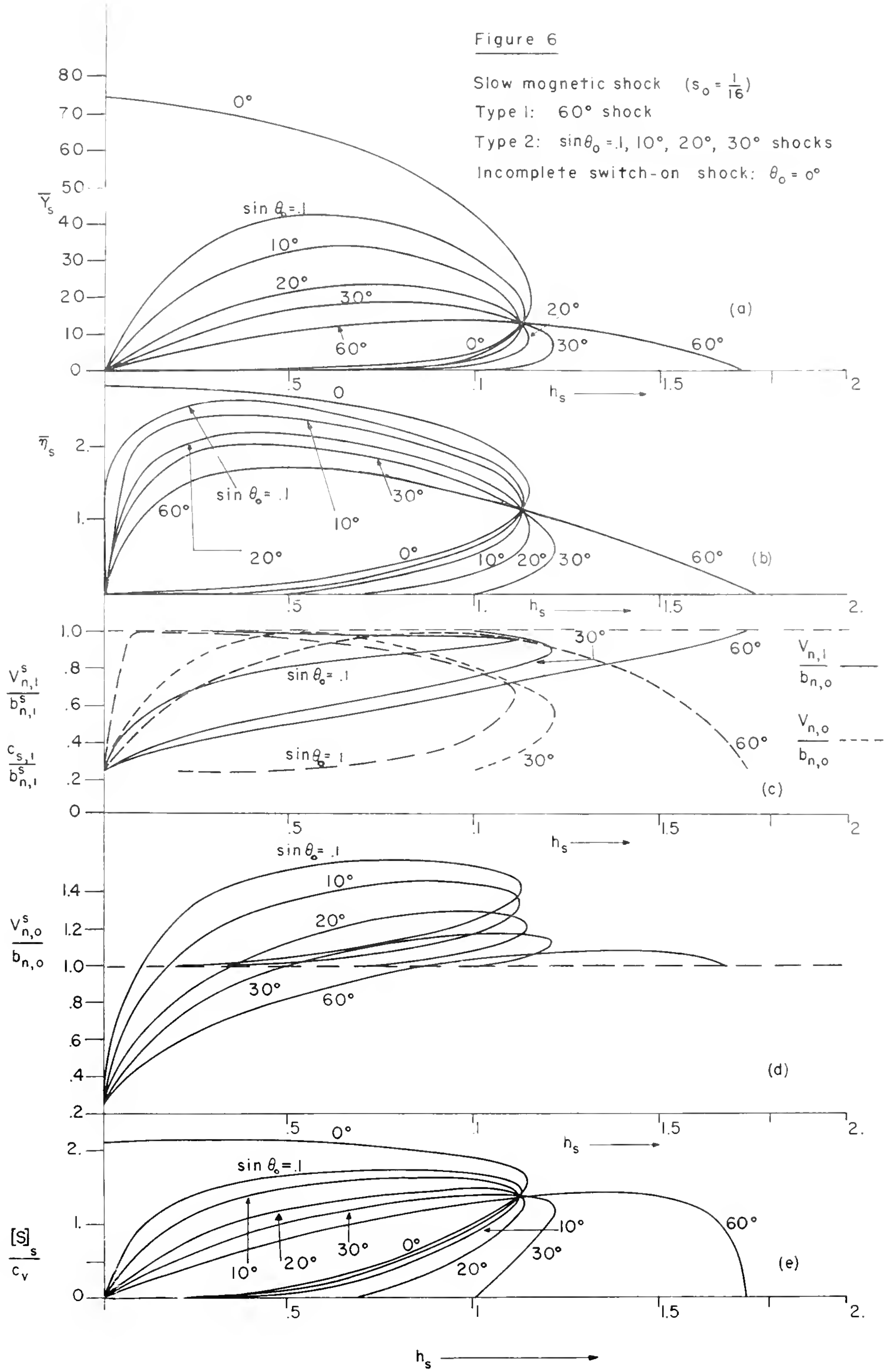
Figure 6
Slow magnetic shock ($s_o = \frac{1}{16}$)
Type 1: 60° shock
Type 2: $\sin\theta_o$ = .1, 10°, 20°, 30° shocks
Incomplete switch-on shock: θ_o = 0°
$\overline{Y}_s$
$\overline{\eta}_s$
$\frac{V^s_{n,1}}{b^s_{n,1}}$
$\frac{c_{s,1}}{b^s_{n,1}}$
$\frac{V_{n,1}}{b_{n,o}}$
$\frac{V_{n,o}}{b_{n,o}}$
$\frac{V^s_{n,o}}{b_{n,o}}$
$\frac{[S]_s}{c_v}$
h_s
(a)
(b)
(c)
(d)
(e)

<u>Figure 6</u>. Illustrating the dependence in M_s shocks of $\bar{Y}_s$, $\bar{\eta}_s$, $V^s_{n,1}/b^s_{n,1}$ $c_{s,1}/b^s_{n,1}$, $V^s_{n,o}/b_{n,o}$, $[S]_s/c_v$ on the shock strength parameter h_s for several values of the parameter Θ_o. The curves corresponding to values of $\theta^o \leq 30^o$ are type 2 curves; 60^o-curves are type 1 curves. The 0^o-curves represent the state behind an incomplete switch-on shock. In type 1 curves, the maximum value of h_s is $\hat{h}_s = 2 \sin \Theta_o$; in type 2 curves the maximum value of h_s is $\hat{\hat{h}}_s$ [see (2.33)]. In the 0^o-limit the curves reduce to those corresponding to the slow gas shock—switch-on shock combination [see Section 2.6.3, also Figure 4]. Those portions of the vertical axis intercepted by the switch-on shock curves correspond to the slow pure gas shock.

Figure 7

Slow magnetic shock

($s_o = \frac{1}{16}$, $\theta_o = 30°$)

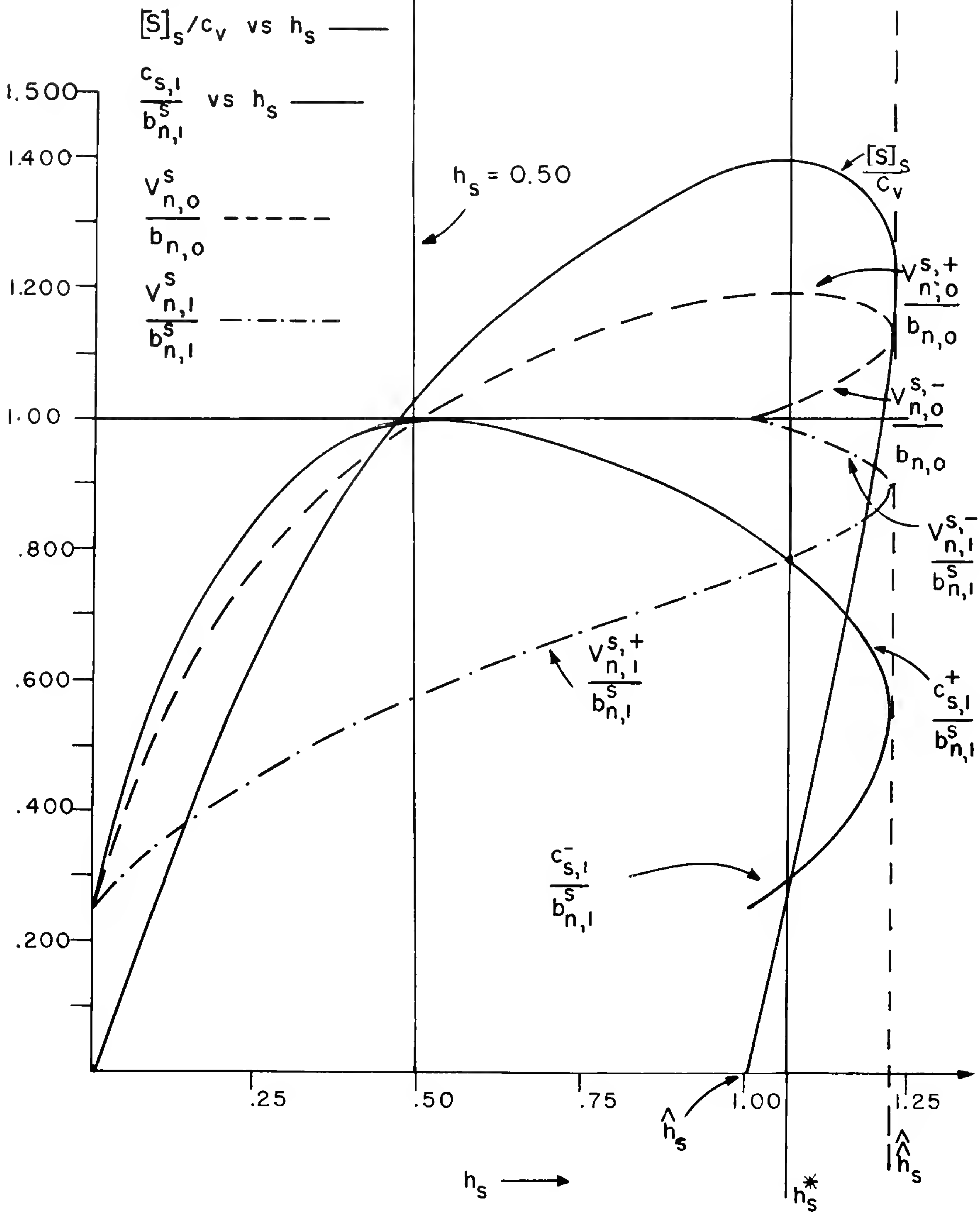

Figure 7. Depicting the variation of $V^s_{n,1}/b^s_{n,1}$, $V^s_{n,o}/b_{n,o}$, $c_{s,1}/b^s_{n,1}$ and $[S]_s/c_v$ with the shock strength parameter h_s. Note that $[S]_s/c_v$ and $V^s_{n,o}/b_{n,o}$ reach their maximum values at the point where $V^s_{n,1} = c_{s,1}$ - i.e., at $h_f = h^*_s$. The plus and minus signs which appear as superscripts (e.g., $V^{s,+}_{n,1}$, $V^{s,-}_{n,1}$, etc.) denote the plus and minus branches of the curves. Note that $V^s_{n,o}/b_{n,o}$ is unity when $h_s = \hat{h}_s = 2 \sin \Theta_o = 1.0$ and when $h_s = \sin \Theta_o = 0.5$.

3. The inadmissibility of expansive shocks - The relation between $d[S]$ and dm in compressive shocks

3.1 The inadmissibility of expansive shocks and related results

In this subsection we shall show that only non-expansive shocks satisfy the basic discontinuity relations $A_1 - A_5$ of (1.3). Our method of proof will consist in showing that i) $[\tau] < 0$ implies $[S] > 0$, ii) $[\tau] = 0$ implies $[S] = 0$ (and conversely) and iii) $[\tau] > 0$ implies $[S] < 0$. As a by-product of our proof of i)-iii) we shall find that $[\tau] < 0$ implies $[p] > 0$ and conversely, and that $[\tau]$ vanishes if, and only if, $[p]$ vanishes. Note that expansive shocks violate (1.3 $-A_5$) because of iii) and hence are inadmissible. Note also that statements i) - iii) are equivalent to the following assertions:

a) The entropy behind a shock exceeds the entropy in front if, and only if, the density behind exceeds the density in front; b) the entropy behind a shock is the same as that in front if, and only if, the density behind is the same as that in front.

In proving these results we shall require only that the caloric equation of state of the medium, namely,

$$p = g(\tau, S), \tag{3.1}$$

satisfies the following conditions:*

$$\left(\frac{\partial g}{\partial \tau}\right)_S < 0, \quad \left(\frac{\partial^2 g}{\partial \tau^2}\right)_S > 0, \quad \left(\frac{\partial g}{\partial S}\right)_\tau > 0 , \tag{3.2}$$

and that

$$\left(\frac{\partial p}{\partial \tau}\right)_e < 0; \tag{3.3}$$

here p is to be regarded as a function of τ and e.

---

* $\left(\frac{\partial X}{\partial Y}\right)_Z$ denotes the partial derivative of $X[= X(Y,Z)]$ with respect to Y, Z being held fixed.

For any value of S, the first two relations of (3.2) require that the function $g(\tau, S)$ is convex downward. The last relation of (3.2) requires that, for constant specific volume, the pressure increases with entropy. In (3.3) it is assumed that, for constant specific internal energy, the pressure decreases as the specific volume increases. Any polytropic ideal gas satisfies (3.2) and (3.3) since

$$\left(\frac{\partial g}{\partial \tau}\right)_S = -\frac{\gamma p}{\tau}, \qquad \left(\frac{\partial^2 g}{\partial \tau^2}\right)_S = \gamma(\gamma-1)\frac{p}{\tau^2} \tag{3.4}$$

$$\left(\frac{\partial g}{\partial S}\right)_\tau = \frac{p}{c_v}, \qquad \left(\frac{\partial p}{\partial \tau}\right)_e = -\frac{p}{\tau},$$

where the constant c_v is the specific heat at constant volume.

We begin our proof of statement i) - that is, $[\tau] < 0$ implies $[S] > 0$, by introducing the hydromagnetic analog of the Hugoniot relation:

$$[e] = -[\tau]\left\{\tilde{p} + \frac{\mu[H]^2}{4}\right\}, \qquad \tilde{p} = (p_1 + p_0)/2 . \tag{3.5}$$

This relation, which was first derived by Lüst[4]-(a), may be obtained from the energy relation A_4 after eliminating the velocity terms by means of $A_1 - A_3$ and cancelling the factor $m \neq 0$.

Let (τ_0, p_0) and (τ_1, p_1) be the specific volume and pressure in front of and behind the shock respectively, e_0 and e_1 the corresponding specific internal energies and

$$E_0: \quad p = h(\tau, e_0) \tag{3.6}$$

and

$$E_1: \quad p = h(\tau, e_1) \tag{3.7}$$

the curves of constant specific internal energy through (τ_0, p_0) and (τ_1, p_1).

Our first step will be to prove that E_1, in a compressive shock, lies above E_o and that p_1 exceeds p_o*. To this end we note, employing (3.5), that

(3.8) $$e_1 - e_o > 0$$

in a compressive shock. Next, we observe that

(3.9) $$\left(\frac{\partial p}{\partial e}\right)_\tau > 0.$$

This relation follows from the identity**

(3.10) $$dp = \left[\left(\frac{\partial g}{\partial \tau}\right)_S + \frac{p}{T}\left(\frac{\partial g}{\partial S}\right)_\tau\right] . d\tau + \frac{1}{T}\left(\frac{\partial g}{\partial S}\right)_\tau de$$

after setting $d\tau = 0$ and making use of the last assumption of (3.2). From (3.8) and (3.9) it follows that E_1 lies above E_o in the (τ, p)-plane. This result and (3.3) then imply that p_1 exceeds p_o. Figure 8, below, depicts the the disposition of E_1 and E_o in the (τ, p)-plane.

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*The proof of the statements that $[p] > 0$ implies $[\tau] < 0$, and that $[p] = 0$, if, and only if, $[\tau] = 0$, will be given at the end of this subsection.

**Note that $dp = \left(\frac{\partial g}{\partial \tau}\right)_S d\tau + \left(\frac{\partial g}{\partial S}\right)_\tau dS$; we arrive at the desired identity after substituting for dS from the basic relation, $TdS = de + pd\tau$.

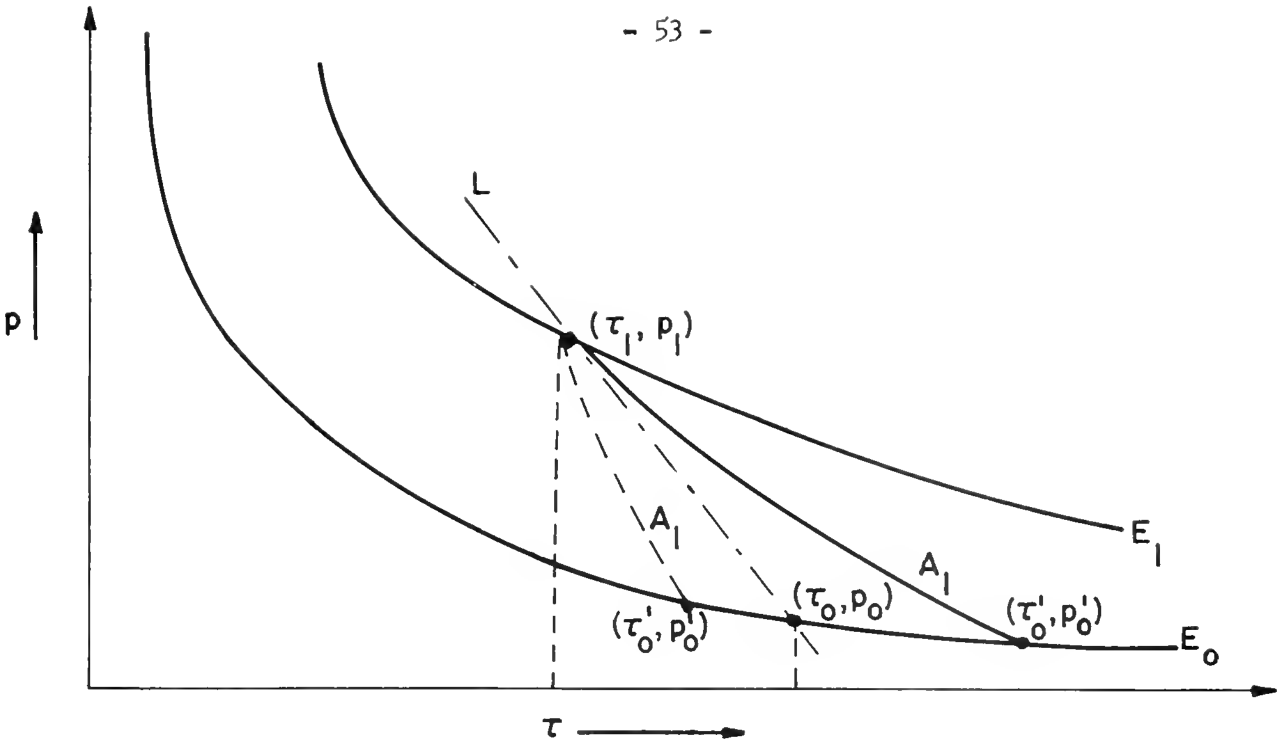

Figure 8: Depicting a means of joining a state (τ_1, p_1) behind a shock to a state (τ_0, p_0) in front by means of an arc of the constant internal energy curve E_0 joining (τ_0, p_0) to (τ_0', p_0'), followed by the arc of the (full-lined) adiabatic curve A_1 joining (τ_0', p_0') and (τ_1, p_1). L is the line through (τ_1, p_1) and (τ_0, p_0). The dashed curve A_1 is the hypothetical adiabatic arc introduced in the proof of case i) and shown to be inadmissible, while E_1 is the curve of constant energy through (τ_1, p_1).

Now let

$$A_1: \quad p = g(\tau, S_1) \tag{3.11}$$

be the adiabatic curve through (τ_1, p_1). Since, from (3.10), (3.2) and (3.3) we have

$$0 > \left(\frac{\partial p}{\partial \tau}\right)_e > \left(\frac{\partial g}{\partial \tau}\right)_S , \tag{3.12}$$

we may conclude that A_1 lies below E_1 as long as $\tau > \tau_1$ [see Figure 8]. Let (τ_0', p_0') be the point of intersection of A_1 with E_0. If ΔS is the change in entropy along the arc of E_0 which joins (τ_0, p_0) and (τ_0', p_0'), then employing the basic relation

(3.13) $\quad TdS = de + pd\tau$

we find, since de vanishes on E_o, that

(3.14) $$\Delta S = \int_{\tau_o}^{\tau_o'} \frac{p}{T} d\tau .$$

This change in entropy is the same as the difference in entropy $[S]$ of the states (τ_1, p_1) and (τ_o, p_o) since the point (τ_o', p_o') is joined to (τ_1, p_1) by the adiabatic path A_1. Thus we conclude

(3.15) $$[S] = \int_{\tau_o}^{\tau_o'} \frac{p}{T} d\tau .$$

The problem of proving $[\tau] < 0$ implies $[S] > 0$ is therefore reduced to showing that τ_o' exceeds τ_o.

That τ_o' must be greater than τ_o is a consequence of the Hugoniot relation (3.5) and our assumption (3.2) that $p = g(\tau, S_1)$ is convex downward. This may be seen from the following argument: Suppose $\tau_o' \leq \tau_o$. Then because the curve A_1 is convex downward [see (3.2)] it intersects the line L through (τ_o, p_o) and (τ_1, p_1) in at most two points - (τ_1, p_1) and possibly (τ_o, p_o). Let

(3.16) $$W = \int_{\tau_1}^{\tau_o'} pd\tau$$

be the area between τ_o and τ_o' and under A_1. This area is clearly less than the area between τ_1 and τ_o and under the line L. Thus we have:

(3.17) $$W = \int_{\tau_1}^{\tau_o'} pd\tau < \frac{p_1 + p_o}{2} (\tau_o - \tau_1).$$

But, since A_1 is an adiabatic path, it follows from (3.13) that $W = e_1 - e_o$ and hence, using (3.17), that

$$(3.18) \qquad e_1 - e_o < \frac{p_1 + p_o}{2} (\tau_o - \tau_1).$$

From (3.5), however, we find easily that

$$(3.19) \qquad e_1 - e_o \geq \frac{p_1 + p_o}{2} (\tau_o - \tau_1)$$

for a compressive shock. We have thus arrived at a contradiction. The assumption that $\tau_o' \leq \tau_o$ is therefore false; τ_o' exceeds τ_o and hence $[S] > 0$ in a compressive shock.

To prove the direct part of statement ii) we make use of the fact that the entropy may be regarded as a function, $S = S(e, \tau)$, say, of e and τ. Since $(e_1 - e_o)$ vanishes when $(\tau_1 - \tau_o)$ vanishes [see (3.5)] it follows that the entropy behind the shock is identical with that in front. Thus $[\tau] = 0$ implies $[S]=0$. The proof of the converse follows the proof of iii).

To prove statement iii), namely that $[\tau] > 0$ implies $[S] < 0$, we proceed as follows. Let us write

$$(3.20) \qquad \begin{array}{ll} \tau_1 = \hat{\tau}_o, & \tau_o = \hat{\tau}_1 \\ S_1 = \hat{S}_o, & S_o = \hat{S}_1. \end{array}$$

In terms of this new notation we have to prove that $[\hat{\tau}] < 0$ implies $[\hat{S}] > 0$. The proof of statement i) now applies to the 'roofed' quantities and furnishes the desired result; namely, that an expansive shock implies a decrease in entropy behind the shock.

To prove the converse of ii) we begin by supposing that $[S] = 0$ implies $[\tau] \gtrless 0$. Employing i) and iii), it then follows that $[S] \gtrless 0$, contrary to our original assumption. Thus $[S] = 0$ implies $[\tau] = 0$.

We have yet to show: 1) $[p] > 0$ implies $[\tau] < 0$; 2) $[p] = 0$ if, and only if, $[\tau] = 0$. Now, the statement that $[p] > 0$ implies $[\tau] < 0$ is equivalent

to the statement that $[\tau] \geq 0$ implies $[p] \leq 0$. Since we have found that $S_1 \leq S_o$ for an expansive shock it follows directly from (3.2) that $p_1 = g(\tau_1, S_1) \leq g(\tau_o, S_o) = p_o$ whenever $\tau_1 \geq \tau_o$; this proves 1).

Let $[\tau] = 0$; then we know that $[S] = 0$. Consequently,

$$p_1 = g(\tau_1, S_1) = g(\tau_o, S_o) = p_o.$$

Let $[p] = 0$ and suppose that $[\tau] \gtrless 0$. It follows that $[S] \gtrless 0$. But this result, using (3.2) again, implies $[p] \gtrless 0$, contrary to our original assumption. Thus $[p] = 0$ implies $[\tau] = 0$, which, together with the previous result, proves 2).

3.2 <u>The relation between $d[S]$ and dm in a compressive shock</u>

The object of this subsection is to prove*:

$$T_1 d[S] = \left\{ [\tau]^2 + \frac{[\tau H_y]^2}{H_n^2} \right\} \text{mdm}, \qquad 0 < \varepsilon \leq \varepsilon_1. \tag{3.21}$$

This relation is valid for all compressive shocks; the second term in the braces vanishes when H_n vanishes or when the shock is a pure gas shock.

We begin our proof by observing that the result of eliminating $[u_n]$

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*Again we are assuming that the transverse magnetic field $\bar{H}_{tr}$ in front of and behind the shock is parallel to the y-axis. This assumption is justified in Section 4.

from* $(1.3 - A_{2,n})$ by means of $(1.3 - A_3)$ is

$$(3.22) \qquad m^2[\tau] + [p + \mu H_y^2/2] = 0.$$

If we eliminate $[\vec{u}_{tr}]$ for $(1.3 - A_{1,tr})$ by means of $(1.3 - A_{2,tr})$ we find, in addition, that

$$(3.23) \qquad m^2[\tau H_y] - \mu H_n^2 [H_y] = 0.$$

Taking the differential of (3.22), (3.23) and of the hydromagnetic Hugoniot relation (3.5), we obtain the following differential relations**:

$$(3.24) \qquad 0 = dm^2[\tau]^2 - [p + \mu H_y^2/2]\, d[\tau] + [\tau]\, d[p + \mu H_y^2/2] ,$$

$$(3.25) \qquad 0 = dm^2[\tau H_y]^2/H_n^2 + \mu[H_y]\, d[\tau H_y] - \mu[\tau H_y]\, d[H_y] ,$$

$$(3.26) \qquad 0 = d[e] + \tilde{p}\, d[\tau] + [\tau]\, d[p]/2 + \mu[H_y]^2\, d[\tau]/4 + \mu d([H_y]^2)[\tau]/4.$$

Employing the relations,

$$(3.27) \qquad T_1\, dS_1 = T_1\, d[S] = de_1 + p_1 d\tau_1 = d[e] + p_1 d[\tau],$$

$$(3.28) \qquad [H_y^2] = [H_y]^2 + 2H_{y,o}\, [H_y],$$

---

* In the following, $A_{j,n}$ and $A_{j,tr}$, $j = 1,2$, are the normal and transverse component equations, respectively, of A_j.

** When $H_n = 0$ or when the shock is a pure gas shock, each term in (3.23) is zero. In these cases, (3.25) reduces to $0 = 0$.

equations (3.26) and (3.24) become

$$(3.29) \qquad 2T_1 d[S] = d[\tau]\left\{[p] - \mu[H_y]^2/2\right\} - [\tau]\, d[p] - \mu([\tau]/2) d([H_y]^2),$$

$$(3.30) \qquad [\tau]^2\, dm^2 = d[\tau]\left\{[p] - \mu[H_y]^2/2\right\} - [\tau] d[p] - \mu[\tau]/2\, d([H_y]^2) - \mu[\tau] H_{y,o} d[H_y]$$

$$+ \mu H_{y,1}\, [H_y]\, d[\tau],$$

while (3.25) becomes

$$(3.31) \qquad dm^2\, [\tau H_y]^2/\mu H_n^2 = H_{y,o}[\tau]\, d[H_y] - H_{y,1}\, [H_y]\, d[\tau].$$

If we subtract (3.31) and (3.30) from (3.29), we obtain the desired result, namely (3.21).

4. Magnetic shocks - preliminary remarks

Magnetic shocks, it will be recalled, are compressive shocks in which $[\vec{H}_{tr}]$ does not vanish. In this section we shall rewrite the shock relations (1.3) in a form most appropriate for dealing with magnetic shocks.

We begin by showing that there is no essential loss of generality in assuming that $\vec{H}_{tr}$ and $\vec{u}_{tr}$ both in front of and behind the shock are parallel to the y-axis. Toward this end we eliminate $[\vec{u}_{tr}]$ from (1.3 $-A_1$) by means of the transverse part of (1.3-A_2) - that is, by means of

$$(4.1) \qquad m[\vec{u}_{tr}] = \mu H_n[\vec{H}_{tr}]$$

to get, assuming $H_n \neq 0$,

$$(4.2) \qquad (m^2\tau_1 - \mu H_n^2)\, \vec{H}_{tr,1} = (m^2\tau_o - \mu H_n^2)\, \vec{H}_{tr,o}, \qquad \tau_1 < \tau_o.$$

We consider two cases: case a), $\overline{H}_{tr,o} = 0$; case b) $\overline{H}_{tr,o} \neq 0$. In case a) the factor $m^2\tau_1 - \mu H_n^2$ must vanish; otherwise $\overline{H}_{tr,1}$ would have to vanish for all values of the shock strength parameter and we would not have a magnetic shock. Assuming, then, that $m^2\tau_1 - \mu H_n^2$ vanishes, it is easy to verify that the shock relations determine only the magnitude of $\vec{H}_{tr,1}$, leaving the direction arbitrary. We are therefore at liberty to choose $\vec{H}_{tr,1}$ parallel to the y-axis. In case (b), it follows directly from (4.2)* that $\vec{H}_{tr,1}$ is collinear with $\vec{H}_{tr,o}$. If we take $\vec{H}_{tr,o}$ parallel to the y-axis, $\vec{H}_{tr,1}$ will also have this property. Assuming that this choice has been made it follows from (4.1) that $u_{z,1} - u_{z,o}$ vanishes. By choosing a suitable (moving) coordinate system, it is clear that $u_{z,1}$ and $u_{z,o}$ may be assumed to vanish.

We have thus far supposed that $H_n \neq 0$. If H_n vanishes we find, employing (1.3-A_1) and (1.3 - $A_{2,tr}$), that

$$(4.3) \qquad \tau_1\vec{H}_{tr,1} = \tau_o \vec{H}_{tr,o}$$

$$[\vec{u}_{tr}] = 0 \quad .$$

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*Since $\tau_1 < \tau_o$ the factors $m^2\tau_o - \mu H_n^2$ and $m^2\tau_1 - \mu H_n^2$ in (4.2) cannot vanish simultaneously. If $m^2\tau_o - \mu H_n^2 \neq 0$, then the right-hand side and hence the left-hand side of (4.2) does not vanish. Thus $m^2\tau_1 - \mu H_n^2 \neq 0$ and $\vec{H}_{tr,1} = \vec{H}_{tr,o}(m^2\tau_o - \mu H_n^2)/(m^2\tau_1 - \mu H_n^2)$. If, on the other hand, $m^2\tau_o - \mu H_n^2 = 0$ then $m^2\tau_1 - \mu H_n^2 \neq 0$ and hence $\vec{H}_{tr,1}$ must vanish; here, too, $\vec{H}_{tr,1}$ may be regarded as collinear with $H_{tr,o}$. Actually, as we shall see later, $m^2\tau_o - \mu H_n^2$ vanishes only for isolated values of the shock strength parameter.

Clearly, there is again no real loss of generality in assuming that $\vec{H}_{tr,j}$ and $\vec{u}_{tr,j}$, $j = 0,1$ are parallel to the y-axis.

If we now set

$$(4.4) \qquad h = (H_{y,1} - H_{y,o})/H_o,$$

and make use of the above result, the shock relations (1.3 A_1-A_3) reduce, using the notation introduced in (1.4), to the following set of relations:

$$(4.5) \qquad \begin{array}{ll} B_{1,y} & m[u_y] = \mu H_o^2 \cos\theta_o h, \\ B_{2,n} & m^2 \tau_1 \bar{\eta} = \mu H_o^2 (X + h \sin\theta_o), \\ B_{2,y} & m^2 \tau_1 (h - \bar{\eta}\sin\theta_o) = \mu H_o^2 \cos^2\theta_o\, h, \\ B_3 & [u_n] = - m \tau_1 \bar{\eta}. \end{array}$$

To complete this set of equations we must add (1.3-A_4) or an equivalent relation. Now it is known[4] that (1.3-A_1-A_3) and A_4 imply the Hugoniot relation (3.5) when $m \neq 0$. By retracing the steps of this derivation it is easy to verify that (1.3-A_1-A_3) and the Hugoniot relation imply (1.3-A_4). The Hugoniot relation (3.5) may therefore be used in place of (1.3-A_4). If we employ the (polytopic) ideal gas relations (1.3'), the Hugoniot relation becomes, in the present notation:

$$(4.5) \qquad B_4 \qquad X = \frac{2\bar{\eta}\, s_o}{2-(\gamma-1)\bar{\eta}} + \frac{h^2}{2-(\gamma-1)\bar{\eta}}.$$

Note that (4.5 - B_2 - B_4) are invariant under all changes of sign of h and of θ_o ($\theta_o \neq 0$) which leave the relations $h \sin\theta_o > 0$ and $h \sin\theta_o < 0$ unchanged, while the expression for $[u_y]$ merely changes sign with h. In analyzing M_f and M_s shocks it is clear that we may, without incurring any

essential loss of generality, restrict the ranges of h and Θ_o as follows:

(4.6) M_f $\quad h > 0, \quad 0 < \Theta_o < 90^\circ,$

(4.7) M_s $\quad h < 0, \quad 0 < \Theta_o < 90^\circ.$

In the next two sections we shall regard (4.6) and (4.7) as the definitions of M_f and M_s shocks respectively.

5. Fast magnetic shocks ($h_f > 0$, $0 < \Theta_o < 90^\circ$); M_f shocks

5.1 Fast magnetic shocks of types 1 and 2; $M_f^{(1)}$ and $M_f^{(2)}$ shocks

Our analysis of M_f shocks is based on the following relations:

$$(5.1) \qquad \frac{[u_n]_f}{b^f_{n,1}} = -\bar{\eta}_f \frac{V^f_{n,1}}{b^f_{n,1}} = -\frac{\bar{\eta}_f}{\left[1-(\bar{\eta}_f/h_f)\sin\Theta_o\right]^{1/2}}, \quad b^f_{n,1} = \sqrt{\mu H_n^2/\rho_1^f},$$

$$(5.2) \qquad \frac{[u_y]_f}{b^f_{n,1}} = \frac{b^f_{n,1}}{V^f_{n,1}}\,\frac{h_f}{\cos\Theta_o} = \frac{h_f}{\cos\Theta_o}\left[1-(\bar{\eta}_f/h_f)\sin\Theta_o\right]^{1/2},$$

$$(5.3) \qquad \left(\frac{V^f_{n,1}}{b^f_{n,1}}\right)^2 = \frac{1}{\bar{\eta}_f}\left(\frac{V^f_{n,o}}{b_{n,o}}\right)^2 = \frac{m_f^2}{(\rho_1^f b^f_{n,1})^2} = \frac{1}{\left[1-(\bar{\eta}_f/h_f)\sin\Theta_o\right]},$$

$$(5.4) \qquad \frac{\bar{\eta}_f/h_f - \sin\Theta_o}{1-(\bar{\eta}_f/h_f)\sin\Theta_o} = \frac{X_f}{h_f} = \frac{\left[\frac{\gamma}{2}\, h_f \sin\Theta_o - (1-s_o) \pm \sqrt{R_X(h_f)}\right]}{2\sin\Theta_o - (\gamma-1)h_f},$$

$$(5.5) \qquad \frac{X_f/h_f + \sin\Theta_o}{1+(X_f/h_f)\sin\Theta_o} = \frac{\bar{\eta}_f}{h_f} = \frac{\left[-\frac{\gamma}{2}\, h_f \sin\Theta_o - (1-s_o) \pm \sqrt{R_{\bar{\eta}}(h_f)}\right]}{2\, s_o \sin\Theta_o - (\gamma-1)h_f},$$

$$(5.6)\qquad \frac{d(X_f/h_f)}{dh_f} = \frac{(\gamma-1)(X_f/h_f)^2 + \gamma \sin\theta_o (X_f/h_f) + 1}{\pm 2\sqrt{R_X(h_f)}},$$

$$(5.7)\qquad \frac{d(X_f/h_f)}{d(\bar{\eta}_f/h_f)} = \frac{\cos^2\theta_o}{(1-(\bar{\eta}_f/h_f)\sin\theta_o)^2},$$

where

$$(5.8)\qquad \text{a)}\quad R_{\bar{\eta}}(h_f) = \left[\frac{\gamma}{2} h_f \sin\theta_o + (1-s_o)\right]^2 + (2s_o \sin\theta_o - (\gamma-1)h_f)(h_f + 2\sin\theta_o),$$

$$\text{b)}\quad R_X(h_f) = \left[\frac{\gamma}{2} h_f \sin\theta_o - (1-s_o)\right]^2 + (2\sin\theta_o - (\gamma-1)h_f)(h_f + 2 s_o \sin\theta_o).$$

It is easy to verify that

$$(5.9)\qquad R_{\bar{\eta}}(h_f) = R_X(h_f) \equiv R(h_f) = h_f^2 \left\{\frac{\gamma^2 \sin^2\theta_o}{4} - (\gamma-1)\right\} + h_f \sin\theta_o (2-\gamma)(1+s_o) + (1-s_o)^2 + 4 s_o \sin^2\theta_o .$$

Equations (5.1)-(5.3) follow almost immediately from ($4.5\text{-}B_{1,y}$, B_3 and $B_{2,y}$) and the fact that $m = \rho_1 V_{n,1} = \rho_o V_{n,o}$. The first equation of (5.4) is obtained by eliminating $m^2 \tau_1$ from ($4.5\text{-}B_{2,n}$) by means of ($4.5\text{-}B_{2,y}$). The first equation of (5.5) is obtained from this result simply by solving for $\bar{\eta}_f/h_f$ in terms of X_f/h_f. The last equations in (5.4) and (5.5) are obtained as follows: When $\bar{\eta}_f/h_f$ is eliminated from the first equation in (5.4) by means of the Hugoniot relation ($4.5\text{-}B_4$), it is found that

$$(5.10)\qquad \left(\frac{X_f}{h_f}\right)^2 \left\{2\sin\theta_o - (\gamma-1)h_f\right\} + \frac{X_f}{h_f}\left\{2(1-s_o) - \gamma h_f \sin\theta_o\right\} - (h_f + 2 s_o \sin\theta_o) = 0.$$

Similarly, employing the Hugoniot relation to eliminate X_f/h_f from the first equation of (5.5) it is found that

$$(5.11)\quad \left(\frac{\bar{\eta}_f}{h_f}\right)^2\left\{2\, s_o \sin\theta_o-(\gamma-1)h_f\right\}+\left(\frac{\bar{\eta}_f}{h_f}\right)\left\{\gamma h_f \sin\theta_o+2(1-s_o)\right\}-(h_f+2\sin\theta_o)=0.$$

The desired results - the last equations in (5.4) and (5.5) - are obtained by solving these equations for X_f/h_f and $\bar{\eta}_f/h_f$. To derive (5.6) we differentiate (5.10) with respect to h_f, solve for $d(X_f/h_f)/dh_f$ and employ the last equation of (5.4) to remove X_f/h from the denominator of the resulting expression. Equation (5.7) follows immediately from the first equation of (5.4) on differentiating X_f/h_f with respect to $\bar{\eta}_f/h_f$.

The expressions for $\bar{\eta}_f/h_f$ and X_f/h_f which result from choosing the plus branch before the radical in (5.4) and (5.5) will be referred to as <u>plus branches</u>* and will be denoted by X_f^+/h_f and $\bar{\eta}_f^+/h_f$; the remaining branches will be called <u>minus</u> branches, and denoted by X_f^-/h_f and $\bar{\eta}_f^-/h_f$. Employing (5.6) and (5.7) we obtain the following useful result: <u>The h_f-derivatives of X/h_f and $\bar{\eta}_f/h_f$ are positive or negative according as we are dealing with the plus or minus branches</u>; it is assumed here that h_f is limited to those values for which $\sqrt{R}$ is real.

Equations (5.1)-(5.3) and the last equations in (5.4) and (5.5) determine a one-parameter family of states behind the shocks, with h_f playing the role of the shock strength parameter.

We turn now to the problem of ascertaining the <u>admissible range of</u> h_f, that is, the totality of values of h_f for which the above one-parameter family is an <u>admissible</u> (see Section 1.2) solution of the shock relations. In light of our definition of admissibility and the results of Section 3, it is clearly sufficient to determine the range (or ranges) of h_f in which a) the state behind the shock is real valued; b) the shock is compressive, i.e., $\bar{\eta}_f > 0$; c) the state behind the shock depends continuously on h_f and the state in front.

Conditions a) and b) are equivalent to the conditions

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*It is easy to verify, using the first equation of (5.4), that $\bar{\eta}_f^+/h_f$ corresponds to X_f^+/h_f and $\bar{\eta}_f^-/h_f$ corresponds to X_f^-/h_f.

$$(5.12)\qquad \begin{aligned} &a')\quad R(h_f) \geq 0 \\ &b')\quad \sin\theta_o < \bar{\eta}_f/h_f < 1/\sin\theta_o . \end{aligned}$$

Condition b') is, in turn, equivalent to the relation

$$(5.13)\qquad b'')\quad X_f/h_f > 0.$$

To prove these equivalences we observe first that if X_f/h_f and $\bar{\eta}_f/h_f$ are real, then a') is valid. Second, we note that X_f/h_f $(= \gamma\bar{Y}_f s_o/h_f + h_f/2)$ is positive; for, $h_f > 0$, and $\bar{\eta}_f > 0$ imply $\bar{Y}_f > 0$. Employing this result and the first equation of (5.4) we find that $X_f/h_f > 0$ if, and only if, the relation (5.12-b') holds. Condition a') and b') are therefore necessary. That a') and b') are sufficient may be concluded from a similar argument. Below, we shall employ a'),b'), or b'')and statement c) above to determine the admissible range of h_f. We begin by rewriting the second equation of (5.4) as follows:

$$(5.14)\qquad \frac{X_f}{h_f} = \frac{b_X \pm \sqrt{R_X}}{c_X} ,$$

where

$$(5.15)\qquad b_X = \frac{\gamma}{2} h_f \sin\theta_o - (1-s_o) ,$$

$$(5.16)\qquad c_X = 2\sin\theta_o - (\gamma-1)h_f ,$$

$$(5.17)\qquad R_X = b_X^2 + (2\sin\theta_o - (\gamma-1)h_f)(h_f + 2 s_o \sin\theta_o).$$

Now let us restrict h_f to the range

$$(5.18)\qquad 0 < h_f < \hat{h}_f = 2\sin\theta_o/(\gamma-1).$$

In this range the product of the last two terms in the right-hand side of (5.17)

is positive; hence $\sqrt{R_X}$ exceeds $|b_X|$. In addition, c_X is positive. Clearly then, only the plus branch leads to positive values for X_f/h_f and conversely. Since $\bar{\eta}_f^+/h_f$ corresponds to X_f^+/h_f [see last footnote], it follows that only the plus branches are admissible in the range $0 < h_f < \hat{h}_f$.

We turn now to the problem of determining under what circumstances it is possible to continue the plus branch into the range

$$(5.19) \qquad h_f \geq \hat{h}_f = 2 \sin \theta_o/(\gamma-1) .$$

To this end let us evaluate the function $\bar{\eta}_f/h_f$ at $h_f = \hat{h}_f$; it is easily verified if we use the second equation of (5.5), (5.9), and (5.17), that

$$(5.20) \qquad \left.\frac{\bar{\eta}_f^+}{h_f}\right|_{h_f=\hat{h}_f} = \begin{cases} 1/\sin \theta_o \quad , & \text{if} \quad b_X\Big|_{h_f = \hat{h}_f} \geq 0, \\ \dfrac{\gamma \sin \theta_o}{(\gamma-1)(1-s_o)} , & \text{if} \quad b_X\Big|_{h_f=\hat{h}_f} < 0. \end{cases}$$

The curves passing through $(1/\sin \theta_o)$ at $h_f = \hat{h}_f$ will be referred to as type 1 curves, and those passing through $\gamma \sin \theta_o/(\gamma-1)(1-s_o)$ at $h_f = \hat{h}_f$ as type 2 curves. These form two mutually exclusive families characterized by the relations

$$(5.21) \qquad \begin{aligned} &\text{type 1:} \quad s_o \geq 1 - \gamma \sin^2\theta_o/(\gamma-1), \\ &\text{type 2:} \quad s_o < 1 - \gamma \sin^2\theta_o/(\gamma-1); \end{aligned}$$

these are simply the conditions $b_X\Big|_{h_f=\hat{h}_f} \geq 0$ and $b_X\Big|_{h_f=\hat{h}_f} < 0$ written out.

Since X_f/h_f becomes infinite when $\bar{\eta}_f^+/h_f$ approaches $1/\sin \theta_o$ [see (5.4)], it is clear that type 1 curves cannot be continued into the region $h_f \geq \hat{h}_f$. For type 2 curves, $\gamma \sin \theta_o/(\gamma-1)(1-s_o)$ [see (5.20)] is clearly positive; (5.12-b') is therefore automatically satisfied. Consequently, using (5.4)

again, X_f^+/h_f is finite and positive. In addition, it follows from (5.6) and (5.7) that $d(\bar{\eta}_f^+/h_f)/dh_f$ is a finite positive value. Clearly, then type 2 curves may be continued into the region $h_f \geq \hat{h}_f$.

Let us determine an upper bound for this range by finding the maximum value for which $R_X \geq 0$, i.e., for which

$$(5.22) \qquad b_X^2 - ((\gamma-1)h_f - 2 \sin \theta_o)(h_f + 2s_o \sin \theta_o) \geq 0.$$

To this end, we observe that in curves of type 2, b_X is zero at a value of h_f, h_f', say, which exceeds $\hat{h}_f$; for b_X vanishes at the value $h_f' = 2(1-s_o)/\gamma \sin \theta_o$ which clearly exceeds $\hat{h}_f = 2 \sin \theta_o/(\gamma-1)$ when $s_o < 1 - \gamma \sin^2\theta_o/(\gamma-1)$ [cf. (5.21), type 2]. In addition, we note that R_X is positive at $h_f = \hat{h}_f$ and decreases monotonically to a negative value at $h_f = h_f'$; for b_X^2 decreases while the two factors in the last term of (5.22) decrease as h_f increases from $\hat{h}_f$ to h_f'. Thus R_X has a zero between these limits. Now, as a mere inspection of the coefficients of (5.9) will reveal, R_X has a positive root if, and only if,

$$\sin^2\theta_o < 4(\gamma-1)/\gamma^2.^*$$

Setting the right-hand side of (5.9) equal to zero we find the following expression for this positive root:

$$(5.23) \qquad \hat{\hat{h}}_f = \frac{\sin \theta_o(2-\gamma)(1+s_o) + 2 \cos \theta_o \sqrt{(\gamma-1)(1-s_o)^2 + s_o\gamma^2\sin^2\theta_o}}{2(\gamma-1) - (\gamma^2\sin^2\theta_o/2)} .$$

Thus $\sqrt{R}$ is real in the range $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$.

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*This relationship is satisfied by a type 2 curve for values of $\gamma < 4$.

Two simple facts are decisive in determining whether, in curves of type 2, the entire range $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$ is admissible. The first is: $\sqrt{R_X} < |b_X|$, $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$. This result follows immediately from (5.22) because one factor in the product term is positive while the other is negative. The second is: $b_X < 0$, $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$. This is immediate for curves of type 2; in fact, b_X is negative in the larger range $\hat{h}_f \leq h_f < h_f'$, where, it will be recalled, h_f' is the zero of b_X.

Now let us consider the expression for X_f/h_f in (5.14). In the range $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$, c_X is negative and, as we have just seen, $\sqrt{R_X} < |b_X|$. Consequently, using (5.14), we find that $X_f/h_f > 0$ in the given range if, and only if, $b_X < 0$.

It follows that X_f/h_f is admissible in the entire range $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$. Moreover, <u>both plus and minus branches of X_f/h_f and hence of $\bar{\eta}_f/h_f$ are admissible</u>. X_f/h_f and $\bar{\eta}_f/h_f$ are thus double-valued functions in the range $\hat{h}_f \leq h_f \leq \hat{\hat{h}}_f$, their minus branches joining the corresponding plus branches at endpoint of the range, namely, at $h_f = \hat{\hat{h}}_f$.

5.2 <u>Description of the curves X_f/h_f versus h_f and $\bar{\eta}_f/h_f$ versus h_f</u>

5.2.1 $M_f^{(1)}$ curves $\left(s_o \geq 1 - \gamma \sin^2\theta_o/(\gamma-1)\right)$

The solid line curves of Figures 9a and 9b depict curves of type 1. As noted above, type 1 curves exist only in the range,

$$(5.24) \qquad 0 < h_f \leq \hat{h}_f = \frac{2 \sin \theta_o}{\gamma-1} ,$$

and within this range only the plus branches are admissible. <u>X_f^+/h_f assumes its minimum value at $h_f = 0$, namely,</u>

$$(5.25) \qquad \left.\frac{X_f^+}{h_f}\right|_{h_f=0} = \frac{\sqrt{(1-s_o)^2 + 4s_o\sin^2\theta_o} - (1-s_o)}{2 \sin \theta_o} ,$$

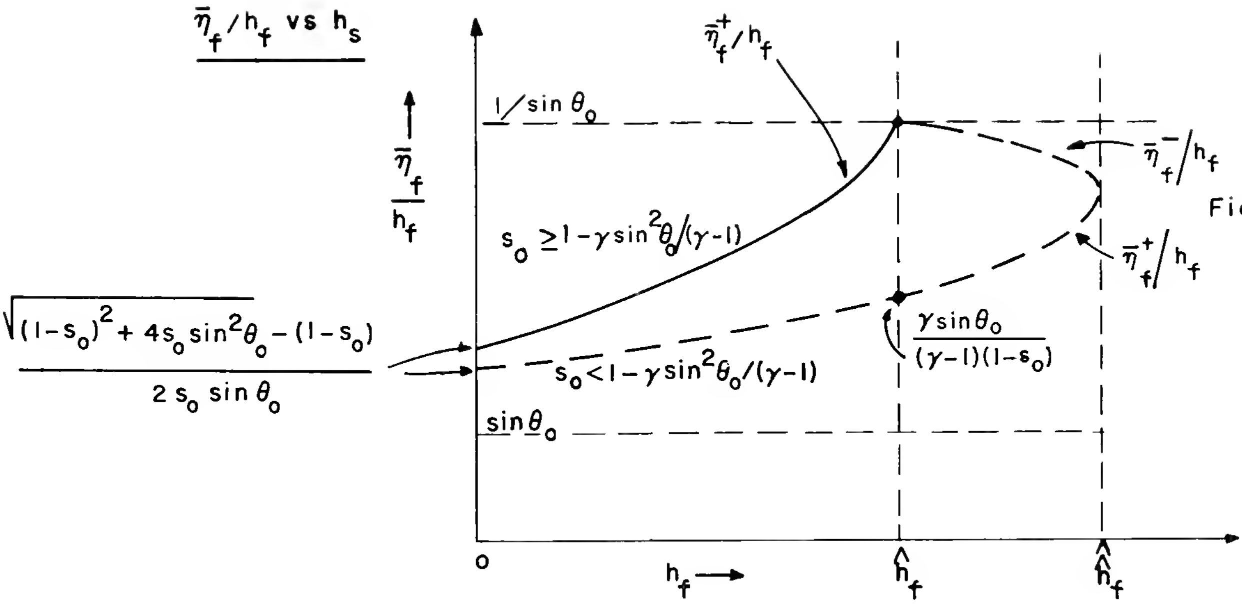

Figure 9(a)

$$\hat{h}_f = 2 \sin\theta_0/(\gamma - 1)$$

$$\hat{\hat{h}}_f = \frac{2\cos\theta_0 \sqrt{(\gamma-1)(1-s_0)^2 + \gamma^2 s_0 \sin\theta_0} + (2-\gamma)(1+s_0)\sin\theta_0}{2(\gamma-1) - \gamma^2 \sin^2\theta_0/2}$$

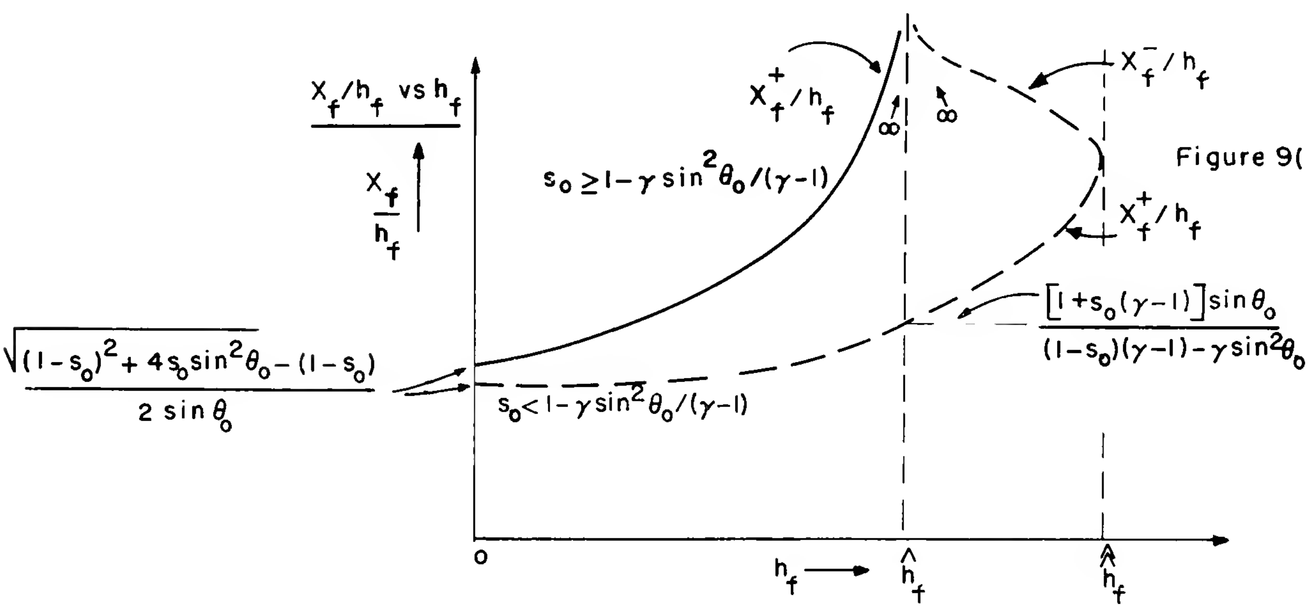

Figure 9(b)

Figures 9(a), 9(b). Sketch of the variation of $\bar{\eta}_f/h_f$ with h_f (a), and of X_f/h_f with h_f, (b) in M_f shocks. The solid line curves are of type 1; the broken line curves, type 2. The (±) superscripts indicate the plus and minus branches. X_f/h_f and $\bar{\eta}_f/h_f$ vary in the same sense as h_f on all plus branches and in the opposite sense to h_f on all minus branches.

and increases monotonically to $+\infty$ [see (5.6), plus branch], approaching the line $h_f = \hat{h}_f$ asymptotically as h_f increases to $\hat{h}_f$. The ratio $\bar{\eta}_f^+/h_f$ varies in the same sense as X_f^+/h_f [see (5.7)]; it increases monotonically from the value

$$(5.26) \qquad \left.\frac{\bar{\eta}_f}{h_f}\right|_{h_f=0} = \frac{\sqrt{(1-s_o)^2 + 4s_o\sin^2\theta_o} - (1-s_o)}{2\, s_o \sin\theta_o},$$

to its maximum value $(1/\sin\theta_o)$ at $h_f = \hat{h}_f$ as h_f increases from zero to $\hat{h}_f$. Note when $h_f = \hat{h}_f$ that $\bar{\eta}_f = 2/(\gamma-1)$ which is the same maximum value achieved in pure gas shocks.

5.2.2 $M_f^{(2)}$ curves $(s_o < 1 - \gamma \sin^2\theta_o/(\gamma-1))$

Curves of type 2 exist in the range

$$(5.27) \qquad 0 < h_f \leq \hat{\hat{h}}_f$$

where $\hat{\hat{h}}_f$ is given in (5.23). In the range

$$(5.27') \qquad 0 < h_f \leq \hat{h}_f = 2\sin\theta_o/(\gamma-1)$$

only the plus branches are admissible; in the range $\hat{h}_f < h_f \leq \hat{\hat{h}}_f$ both plus and minus branches are admissible. Since $c_X < 0$ in the range $\hat{h}_f < h_f \leq \hat{\hat{h}}_f$, the values $(X_f^-/h_f,\ \bar{\eta}_f^-/h_f)$ exceed the corresponding values on the plus branch except at $h_f = \hat{\hat{h}}_f$ where the branches join. Since $d(X_f^{\pm}/h_f)/dh_f$ and $d(\bar{\eta}_f^{\pm}/h_f)/dh_f$ are all infinite at $h_f = \hat{\hat{h}}_f$, it is clear that this joining is a smooth one. X_f^-/h_f and $\bar{\eta}_f^-/h_f$ assume minimum values at $h_f = \hat{\hat{h}}_f$ and increase monotonically [see (5.6) and (5.7)] to $+\infty$ and $(1/\sin\theta_o)$, respectively, as h_f decreases from $\hat{\hat{h}}_f$ to $\hat{h}_f$. On the plus branches, $\bar{\eta}_f^+/h_f$ and X_f^+/h_f assume minimum values at $h_f = 0$ [see (5.25) and (5.26)] and increase montonically [see (5.6) and (5.7)] to maximum values at $h_f = \hat{\hat{h}}_f$. Note that the zeros of the denominators in the

expressions for $\bar{\eta}_f^{\,+}/h$ and X_f^+/h occur at the zeros of the corresponding numerators. These zeros therefore do not produce infinities: in fact, employing L'hospital's rule or solving directly from (5.10) and (5.11), we find that

(5.28)

$$\text{a)}\quad \left.\frac{\bar{\eta}_f^{\,+}}{h_f}\right|_{h_f=\frac{2}{\gamma-1}s_o\sin\theta_o} = \frac{\sin\theta_o\left[s_o/\gamma-1)+1\right]}{1-s_o+\gamma/\gamma-1\ s_o\sin^2\theta_o}$$

$$\text{b)}\quad \left.\frac{X_f^+}{h_f}\right|_{h_f=\frac{2}{\gamma-1}\sin\theta_o} = \frac{\sin\theta_o\left[s_o+1/\gamma-1\right]}{1-s_o-(\gamma/\gamma-1)\sin^2\theta_o}\ .$$

The dashed curves of Figures 9(a) and 9(b) depict typical curves of type 2.

5.3 <u>Some general features of</u> M_f <u>shocks</u>

Our chief object here is i) to prove that $\bar{\eta}_f$, $\bar{Y}_f$, $V^f_{n,1}/b_{n,1}$, $V^f_{n,o}/b_{n,o}$, m_f <u>and</u> $[S]_f$ <u>vary in the same sense, and</u> ii) <u>to determine how these quantities vary with</u> h_f. In view of (5.3) and (3.21), i) will be proved if we can show that $\bar{\eta}_f$ varies in the same sense as $\bar{\eta}_f/h_f$ and $\bar{Y}_f$ varies in the same sense as $\bar{\eta}_f$.

Consider first the variation of $\bar{\eta}_f$ with $\bar{\eta}_f/h_f$. As we showed in the preceding subsection, $\bar{\eta}_f^+/h_f$ varies directly with h_f; this statement is independent of type. Thus $\bar{\eta}_f^+ = (\bar{\eta}_f^+/h_f)h_f$ varies directly with $\bar{\eta}_f^+/h_f$. It remains for us to show that the same is true for the minus branch. Our procedure will be to prove that $d\bar{\eta}_f^-/dh_f$ has the same sign as $d(\bar{\eta}_f^-/h_f)/dh_f$, which, in virtue of (5.6) and (5.7) is negative. Now, evidently we have

$$(5.29)\qquad \frac{1}{h_f}\frac{d\bar{\eta}_f^-}{dh_f} = \frac{d(\bar{\eta}_f^-/h_f)}{dh_f} + \frac{1}{h_f^2}\bar{\eta}_f^-\ .$$

Since $d(\bar{\eta}_f^-/h_f)/dh_f$ is a large negative number in the vicinity of $h_f = \hat{\hat{h}}_f$, it follows that $d\bar{\eta}_f^-/dh_f$ is also a large negative number in this vicinity. To obtain the desired result, it is sufficient to prove that $\bar{\eta}_f^-/h_f$ exceeds $(\sin\theta_o)^{-1}$ when $d\bar{\eta}_f^-/dh_f$ vanishes; for then $d\bar{\eta}_f^-/dh_f$ would be negative through the admissible range. To this end we observe that

$$(5.30) \qquad \frac{d(\bar{\eta}_f^- /h_f)}{dh_f} = -(\bar{\eta}_f^-/h_f)/h_f$$

when $d\bar{\eta}_f^-/dh_f$ vanishes. Differentiating (5.11) with respect to h_f and eliminating $d(\bar{\eta}_f/h_f)/dh_f$ by means of (5.28), we find

$$(5.31) \qquad \left(\frac{\bar{\eta}_f^-}{h_f}\right)^2 \left[4s_o \sin\theta_o - (\gamma-1)h_f\right] = -2(1-s_c)\left(\frac{\bar{\eta}_f^-}{h_f}\right) - h_f$$

The addition of the left-hand side of (5.11) to the right-hand side of (5.31) leads directly to

$$(5.32) \quad \left(\frac{\bar{\eta}_f^-}{h_f}\right)^2 = \left(\frac{\bar{\eta}_f^-}{h_f}\right)\frac{\gamma h_f}{2s_o} - \frac{h_f + \sin\theta_o}{s_o \sin\theta_o} \quad .$$

The requirement that the right-hand side of (5.32) be positive results in the desired relation, namely,

$$\frac{\bar{\eta}_f^-}{h_f} > \frac{2}{\gamma \sin\theta_o}\left(1 + \frac{\sin\theta_o}{h_f}\right) > (\sin\theta_o)^{-1}.$$

Our aim now is to prove that $\bar{Y}_f$ varies directly with $\bar{\eta}_f$. As an inspection of the Hugoniot relation ($4.5 - B_4$) will reveal, $\bar{Y}_f$ increases with increasing $\bar{\eta}_f$ as long as we restrict ourselves to the plus branches of $\bar{Y}_f$ and

$\bar{\eta}_f$; this is so because $\bar{\eta}_f^+$ varies directly with h_f. On the minus branches of $M_f^{(2)}$ shocks the behavior of the second term is no longer decidable by a mere inspection; for here $\bar{\eta}_f$ and h_f vary in opposite senses. Nevertheless, $\bar{Y}_f$ varies directly with $\bar{\eta}_f$ on this branch also, as we shall now show. First, we observe that the $\bar{\eta}_f^-$ derivative of X_f^- is, using the first equation of (5.4),

$$(5.33)\qquad \frac{dX_f^-}{d\bar{\eta}_f^-} = \frac{\left[1-\frac{\bar{\eta}_f^-}{h_f}\sin\theta_o\right]\left[1-\sin\theta_o/(d\bar{\eta}_f^-/dh_f)\right] + \left[\bar{\eta}_f^- - h_f\sin\theta_o\right]\sin\theta_o\, d(\bar{\eta}_f^-/h_f)/d\bar{\eta}_f^-}{\left[1-(\bar{\eta}_f^-/h_f)\sin\theta_o\right]^2}.$$

The right-hand side of this equation is positive, for $(\sin\theta_o)^{-1} > \bar{\eta}_f^-/h_f > \sin\theta_o$ [cf. (5.12-b')] and, as we have seen above, $d(\bar{\eta}_f^-/dh_f)$ and $d(\bar{\eta}_f^-/h_f)/dh_f$ are both negative. The relationship between the derivatives $d\bar{Y}_f^-/d\bar{\eta}_f^-$ and $dX_f^-/d\bar{\eta}_f^-$ may readily be obtained by differentiating (1.4-e). The result is found to be

$$(5.34)\qquad \frac{d\bar{Y}_f^-}{d\bar{\eta}_f^-} = \frac{\gamma}{s_o}\left\{\frac{dX_f^-}{d\bar{\eta}_f^-} - \frac{h_f}{d\bar{\eta}_f^-/dh_f}\right\}.$$

Since $dX_f^-/d\bar{\eta}_f^-$ is positive and $d\bar{\eta}_f^-/dh_f$ is negative, $d\bar{Y}_f^-/d\bar{\eta}_f^-$ exceeds zero, which is the desired result.

It is now a relatively simple matter to prove ii), that is, to determine how $\bar{\eta}_f$, $\bar{Y}_f$, etc., vary with h_f. To this end we show that a) $\bar{\eta}_f$ <u>varies with</u> h_f <u>in the same way that</u> $\bar{\eta}_f/h_f$ <u>varies with</u> h_f [see previous subsection and Figure 9(a)], and b) <u>that</u> $\bar{Y}_f$, $V_{n,1}^f/b_{n,1}$, $V_{n,o}^f/b_{n,o}$, m_f <u>and</u> $[S]_f$ <u>vary with</u> h_f <u>in the same way that</u> X_f/h_f <u>varies with</u> h_f [see Figure 9(b)]. Note that statement a) involves variables of finite range, statement b) of infinite range; the variation with h_f of both types of quantities is the same except for this fact.

To prove a) we merely recall that $\bar{\eta}_f$ varies in the same sense as $\bar{\eta}_f/h_f$. To prove b) we note that $d(X_f/h_f)/d(\bar{\eta}_f/h_f)$ exceeds zero [cf. (5.7)] so that $\bar{Y}_f$, $V_{n,1}^f/b_{n,1}^f$ etc. vary directly with X_f/h_f.

Finally, we would like to show that $\bar{\eta}_f$ and $\bar{Y}_f$ are independent of θ_o when $h_f = 2(1-s_o)/\gamma$, $s_o < 1$. To this end we note $\bar{\eta}_f/h_f = 1$ when $X_f/h_f = 1$ [see (5.5)]. Substituting for h_f or $\bar{\eta}_f$ and X_f in (4.5-B_4) and solving for h_f we find that

(5.35) $$h_f = 2(1-s_o)/\gamma\,, \qquad s_o < 1,$$

(5.36) $$\bar{\eta}_f = 2(1-s_o)/\gamma, \qquad s_o < 1,$$

and, using (1.4-e), that

(5.37) $$\bar{Y}_f = \frac{2(1-s_o)}{\gamma s_o}(\gamma + s_o - 1), \qquad s_o < 1.$$

5.4 Weak fast shocks

If the expression for $\bar{\eta}_f^+$ in (5.5) is expanded in the neighborhood of $h_f = 0$, it is found, after carrying out suitable algebraic manipulations, that

(5.38) $$\bar{\eta}_f = \frac{1-r_f}{\sin\theta_o} h_f + \frac{1}{2s_o}\left[\frac{(\gamma-1)(1-r_f)}{\sin^2\theta_o} - \frac{(\gamma-1)(1+s_o) - \gamma s_o r_f}{1 + s_o - 2s_o r_f}\right] h_f^2 + \ldots$$

where

(5.39) $$\frac{1}{r_f} = \frac{1 + s_o + \sqrt{(1+s_o)^2 - 4s_o\cos^2\theta_o}}{2\cos^2\theta_o} = \frac{c_{f,0}^2}{b_{n,0}^2} > 1,$$

[cf. (1.5)].

A sufficient condition for the convergence of the above expansion is

$$\left\{ \frac{h_f}{\sin\theta_o} \, , \, \frac{h_f}{\sqrt{(1-s_o)^2 + 4s_o \sin^2\theta_o}} \right\} << 1, \quad \theta_o > 0. \tag{5.40}$$

If we invert the power series above to obtain h_f as a function of $\bar{\eta}_f$ and substitute the resulting expression for h_f into equations $B_{1,y} - B_3$ of (4.5), we find, after expanding with respect to $\bar{\eta}_f$ and retaining only the leading terms, that

$$\begin{aligned}
&C_1 \qquad h_f = \frac{\sin\theta_o}{1-r_f}\,\bar{\eta}_f + \cdots , \\
&C_{1,y} \quad [u_y]_f = \frac{r_f c_{f,0} \tan\theta_o}{1-r_f}\,\bar{\eta}_f + \cdots , \\
&C_2 \qquad m_f \tau_1 = V_{n,1} = c_{f,0} + \cdots , \\
&C_3 \qquad [u_n]_f = -c_{f,0}\,\bar{\eta}_f + \cdots , \qquad 0 < \theta_o < 90^\circ .
\end{aligned} \tag{5.41}$$

In addition, C_1 and (2.70) yield the relation

$$\bar{Y}_f = \gamma\bar{\eta}_f + \frac{\gamma(\gamma-1)}{2}\bar{\eta}_f^2 + \frac{\gamma(\gamma-1)^2}{4}\bar{\eta}_f^3 + \frac{\gamma(\gamma-1)\sin^2\theta_o}{4s_o(1-r_f)^2}\bar{\eta}_f^3 + O(\bar{\eta}_f^4) \tag{5.42}$$

which is the same, up to terms of the second order, as the corresponding expansion in a gas shock. The fourth term in (5.42) represents the lowest order contribution arising from the presence of the magnetic field.

It is well known that the increase in entropy behind a pure gas shock is of the third order in the shock strength parameter. A similar statement may be made for a fast magnetic shock. Now the increase in entropy of an ideal polytropic gas which passes from the state (τ_o, p_o) to the state (τ_1, p_1) is given by

$$(5.43) \qquad [S]_f = c_v \log\left[\frac{p_1}{p_o}\left(\frac{\rho_1}{\rho_o}\right)^{-\gamma}\right] = c_v\left\{\log(\overline{Y}_f + 1) - \gamma\log(\overline{\eta}_f + 1)\right\}.$$

Substituting (5.42) into the first term in (5.43) and expanding the resulting expression in the neighborhood of $\overline{\eta}_f = 0$, we find

$$(5.44) \qquad [S]_f = c_v\frac{\gamma(\gamma-1)}{4}\left\{\frac{\gamma+1}{3} + \frac{\sin^2\theta_o}{(1-r_f)^2 s_o}\right\}\overline{\eta}_f^3 + O(\overline{\eta}_f^4) \quad , \qquad \theta_o > 0.$$

The second term in the brackets represents the contribution of the magnetic field; when this term is absent (5.44) reduces to the corresponding expression for a weak pure gas shock.

5.5 Relation of the normal flow velocity (relative to the shock) in front of and behind fast shocks to the disturbance speeds in these regions

Employing (5.3) and the second equation of (5.5), we find

$$(5.45) \qquad \lim_{h_f\to 0} V^f_{n,1} = \lim_{h_f\to 0} V^f_{n,o} = b_{n,o}\left\{\frac{1+s_o+\sqrt{(1+s_o)^2 - 4s_o\cos^2\theta_o}}{2\cos^2\theta_o}\right\}^{1/2} .$$

Comparing the last expression on the right with (1.5) we find that it is identical with the fast disturbance speed $c_{f,o}$ in front of the shock - i.e.,

$$(5.46) \qquad \lim_{h_f\to 0} V^f_{n,1} = \lim_{h_f\to 0} V^f_{n,o} = c_{f,o}.$$

In subsection (5.3) we saw that $V^f_{n,o}/b_{n,o}$ increased with increasing $\overline{\eta}_f$; clearly

$$(5.47) \qquad V^f_{n,o} > c_{f,o}, \qquad 0 < \varepsilon \leq \varepsilon_1.$$

In the region behind the shock it follows directly from (5.3) that

(5.48) $\quad V^f_{n,1} > b^f_{n,1}\ , \qquad 0 < \varepsilon \leq \varepsilon_1.$

We shall now show that

(5.49) $\quad V^f_{n,1} < c^f_{f,1}\ , \qquad 0 < \varepsilon \leq \varepsilon_1.$

As a first step in this direction we observe that (3.24) may be rewritten as follows*:

(5.50) $$V^2_{n,1} = \frac{dp_1}{d\rho_1} + \mu H_{y,1} \frac{dH_{y,1}}{d\rho_1} + [\tau] \frac{dm^2}{d\rho_1}$$

where we have made use of the relation (3.22) and the fact that ${V^f_{n,1}}^2 = m_f^2/\rho_1^2$. If

(5.51) $$p_1 = g(\rho_1, S_1)$$

is the equation of state for the medium behind the shock, then

(5.52) $$\frac{dp_1}{d\rho_1} = \frac{d}{d\rho_1} g\big(\rho_1, S_1(\rho_1)\big) = a_1{}^2 + \frac{\partial g}{\partial S_1} \frac{dS_1}{d\rho_1}$$

which, in virtue of (5.43) and the fact that $c_v = R/(\gamma-1)$, becomes:

(5.53) $$\frac{dp_1}{d\rho_1} = a_1{}^2 + \frac{\gamma-1}{R}\, p_1 \frac{dS_1}{d\rho_1}\ .$$

Thus we find that

(5.54) $$V^2_{n,1} = a_1{}^2 + \mu H_{y,1} H_o \frac{dh}{d\rho_1} + [\tau] \frac{dm^2}{d\rho_1} + \frac{(\gamma-1)}{R}\, p_1 \frac{dS_1}{d\rho_1}\ .$$

This equation may be recast into a most convenient form by making use of the following relations:

*In the proof which follows we shall omit the sub and superscript 'f'.

$$(5.55)\qquad \frac{dh}{d\rho_1} = \frac{H_{y,1}}{H_{y,o}\rho_1}\left(\frac{h}{\bar{\eta}}\right) + \frac{[\tau H_y]^2}{\mu H_{y,o} H_o H_n^2 [\tau]} \frac{dm^2}{d\rho_1},$$

$$(5.56)\qquad \frac{h}{\bar{\eta}} = \frac{\sin\theta_o\, V_{n,1}^2}{V_{n,1}^2 - b_{n,1}^2},$$

$$(5.57)\qquad \frac{dS}{d\rho_1} = \frac{R\rho_1}{2p_1}\left\{[\tau]^2 + \frac{[\tau H_y]^2}{H_n^2}\right\}\frac{dm^2}{d\rho_1}.$$

Equation (5.55) is simply (3.31) rewritten in a form most suited to the present analysis; equation (5.57) follows immediately from (3.21) on making use of the ideal gas relation $p_1 \tau_1 = RT_1$; (5.56) follows directly from the first and last expressions in (5.3) after solving for $h/\bar{\eta}$.

These relations and (5.54) yield

$$(5.58)\qquad V_{n,1}^4 - (a_1^2 + \mu H_1^2 \rho_1^{-1})V_{n,1}^2 + b_{n,1}^2 a_1^2 + (V_{n,1}^2 - b_{n,1}^2)\frac{dm^2}{d\rho_1} G_1 = 0,$$

where

$$(5.59)\qquad G_1 = G_1(\rho_1) \equiv -\frac{[\tau H_y]^2}{[\tau]H_n^2}\frac{[H_y]}{H_{y,o}} - \frac{1}{[\tau]}\left\{[\tau]^2 + \frac{[\tau H_y]^2}{H_n^2}\right\}\left\{1 - \frac{\gamma-1}{2}\bar{\eta}\right\}.$$

It is important to note that G_1 is a positive quantity in fast magnetic shocks where $[\tau] < 0, [H_y] > 0$ and $\bar{\eta} < 2/(\gamma-1)$. Solving for $V_{n,1}^2$ in (5.58) we find that

$$(5.60)\qquad V_{n,1}^2 = \frac{a_1^2 + \mu H_1^2 \rho_1^{-1} + \sqrt{(a_1^2 + \mu H_1^2\rho_1^{-1})^2 - 4b_{n,1}^2 a_1^2 - 4(V_{n,1}^2 - b_{n,1}^2)G_1\frac{dm^2}{d\rho_1}}}{2}$$

where the choice of the plus sign before the radical is dictated by the requirement that $V_{n,1}$ approach $c_{f,o}$ when ρ_1 approaches ρ_o. Employing the fact that $V_{n,1} > b_{n,1}$ in M_f shocks and the fact that $G_1 > 0$, we find

$$(5.61)\qquad V_{n,1}^2 < \frac{a_1^2 + \mu H_1^2\rho_1^{-1} + \sqrt{(a_1^2 + \mu H_1^2\rho_1^{-1})^2 - 4a_1^2\, b_{n,1}^2}}{2}\ , \quad \rho_1 > \rho_o\ .$$

It is now a simple matter to verify that the right-hand side of (5.61) is $c_{f,1}^2$; all that is involved is a change of notation [cf. (1.4) and (1.5)].

6. Slow magnetic shocks ($h_s > 0$, $0 < \theta_o < 90^o$); M_s shocks

6.1 Slow magnetic shocks of types 1 and 2; $M_s^{(1)}$ and $M_s^{(2)}$ shocks

Our analysis of M_s shocks is based on the following relations:

$$(6.1)\qquad \frac{[u_n]_s}{b_{n,1}^s} = -\bar{\eta}_s\,\frac{V_{n,1}^s}{b_{n,1}^s} = -\frac{-\bar{\eta}_s}{\left[1+(\bar{\eta}_s/h_s)\sin\theta_o\right]^{1/2}}\ , \quad b_{n,1}^s = \sqrt{\mu H_n^2/\rho_1^s}\ ,$$

$$(6.2)\qquad \frac{[u_y]_s}{b_{n,1}^s} = -\frac{b_{n,1}^s}{V_{n,1}^s}\,\frac{h_s}{\cos\theta_o} = -\frac{h_s}{\cos\theta_o}\left[1+(\bar{\eta}_s/h_s)\sin\theta_o\right]^{1/2}\ ,$$

$$(6.3)\qquad \left(\frac{V_{n,1}^s}{b_{n,1}^s}\right)^2 = \frac{1}{\bar{\eta}_s}\left(\frac{V_{n,o}^s}{b_{n,o}}\right)^2 = \frac{m_s^2}{(\rho_1^s b_{n,1}^s)^2} = \frac{1}{\left[1+(\bar{\eta}_s/h_s)\sin\theta_o\right]}\ ,$$

$$(6.4)\qquad \frac{\bar{\eta}_s/h_s + \sin\theta_o}{1+(\bar{\eta}_s/h_s)\sin\theta_o} = \frac{X_s}{h_s} = \frac{\left[\frac{\gamma}{2}\, h_s\sin\theta_o + (1-s_o) \pm \sqrt{\hat{R}_X(h_s)}\right]}{2\sin\theta_o + (\gamma-1)h_s}$$

$$(6.5)\qquad \frac{X_s/h_s - \sin\theta_o}{1-(X_s/h_s)\sin\theta_o} = \frac{\bar{\eta}_s}{h_s} = \frac{\left[(1-s_o) - \frac{\gamma}{2}\, h_s\sin\theta_o \pm \sqrt{\hat{R}_{\bar{\eta}}(h_s)}\right]}{2\, s_o\sin\theta_o + (\gamma-1)h_s}$$

$$(6.6)\qquad \frac{d(\bar{\eta}_s/h_s)}{dh_s} = \frac{-\left[(\gamma-1)(\bar{\eta}_s/h_s)^2 + \gamma\sin\theta_o\,\bar{\eta}_s/h_s + 1\right]}{\pm\sqrt{\hat{R}_{\bar{\eta}}(h_s)}}$$

$$(6.7) \qquad \frac{d(X_s/h_s)}{d(\bar{\eta}_s/h_s)} = \frac{\cos^2\theta_o}{(1+(\eta_s/h_s)\sin\theta_o)^2}$$

where

$$(6.8) \qquad \text{a)} \quad \hat{R}_{\bar{\eta}}(h_s) = \left[(1-s_o) - \frac{\gamma}{2} h_s \sin\theta_o\right]^2 - (2s_o \sin\theta_o + (\gamma-1)h_s)(h_s - 2\sin\theta_o)$$

$$\text{b)} \quad \hat{R}_X(h_s) = \left[\frac{\gamma}{2} h_s \sin\theta_o + (1-s_o)\right]^2 - (2\sin\theta_o + (\gamma-1)h_s)(h_s - 2s_o \sin\theta_o)$$

and

$$(6.9) \qquad \hat{R}_{\bar{\eta}}(h_s) = \hat{R}_X(h_s) = h_s^2\left\{\frac{\gamma^2\sin^2\theta_o}{4} - (\gamma-1)\right\} - h_s \sin\theta_o(2-\gamma)(1+s_o)$$

$$+ (1-s_o)^2 + 4s_o\sin^2\theta_o .$$

The simplest way to derive (6.1)-(6.5) and (6.7), (6.8) is to recognize that (5.1)-(5.5) and (5.7), (5.8) are solutions of the shock relations for negative as well as for positive values of h_f. If we set

$$(6.10) \qquad h_s = -h_f = \frac{H_{y,o} - H_{y,1}}{H_o}, \qquad h_f < 0,$$

and substitute $(-h_s)$ into (5.1)-(5.5), (5.7), (5.8) we obtain the corresponding formulas (6.1)-(6.5), (6.7), (6.8). To derive (6.6) we substitute $(-h_s)$ for h_f in (5.11), differentiate with respect to h_s, solve for $\bar{\eta}_s/h_s$ and employ the last equation of (6.5) to remove $\bar{\eta}_s/h_s$ from the denominator of the resulting expression.

According as the plus and minus sign appears before the radical in (6.4) and (6.5), we shall speak of the plus or minus branch of X_s/h_s and $\bar{\eta}_s/h_s$ and of derived quantities. As before, the plus branches of $\bar{\eta}_s/h_s$ and X_s/h_s will be denoted by η_s^+/h_s and X_s^+/h_s, the minus branches by η_s^-/h_s, X_s^-/h_s. It is easy to verify, using the first equation in (6.4), that $\bar{\eta}_s^+/h_s$ corresponds to X_s^+/h_s

and $\bar{\eta}_s^-/h_s$ to X_s^-/h_s. From (6.6) and (6.7) we obtain the following useful result: The h_s derivatives of $\bar{\eta}_s/h_s$ and X_s/h_s are negative or positive according as we are dealing with the plus or minus branches; it is assumed here that h_s is limited to those values for which $\sqrt{\hat{R}(h_s)}$ is real.

Employing a procedure similar to that used in determining admissible fast shock solutions, we find that admissible slow shocks may also be conveniently classified into two types. Slow shocks of type 1 are those in which $s_o \geq 1 - \gamma \sin^2\theta_o$; slow shocks of type 2 are those in which $s_o < 1 - \gamma \sin^2\theta_o$. In the following we shall give a brief description, based on the second equations of (6.4), (6.5) and the remark underlined above, of the $\bar{\eta}_s/h_s$ vs. h_s and X_s/h_s vs. h_s curves of types 1 and 2.

6.2 Description of the curves $\bar{\eta}_s/h_s$ versus h_s and X_s/h_s versus h_s

6.2.1 $M_s^{(1)}$ curves $(s_o \geq 1 - \gamma \sin^2\theta_o)$

Here, admissible M_s shocks exist only in the range

$$0 < h_s \leq \hat{h}_s = 2 \sin\theta_o .$$

Within this range, moreover, only the plus branch $\bar{\eta}_s^+/h_s$ is admissible. As h_s increases from zero to $\hat{h}_s$, $\bar{\eta}_s^+/h_s$ decreases monotonically [see (6.6)] from the finite value

$$\left.\bar{\eta}_s^+/h_s\right|_{h_s = 0} = \frac{(1-s_o) + \sqrt{(1-s_o)^2 + 4s_o\sin^2\theta_o}}{2s_o\sin\theta_o}, \tag{6.11}$$

to zero [see (6.5)]. Note that $H_{y,1}$ decreases from $H_{y,0}$ to $-H_{y,0}$ as h_s increases from zero to $2\sin\theta_o$. A typical $(\bar{\eta}_s^+/h_s)$ vs. h_s-curve of type 1 is illustrated by the solid curve of Figure 10(a). The behavior of X_s^+/h_s may be ascertained from the behavior of $\bar{\eta}_s^+/h_s$ using equations (6.4) and (6.7). In particular, from (6.7) it follows that X_s^+/h_s varies in the same sense

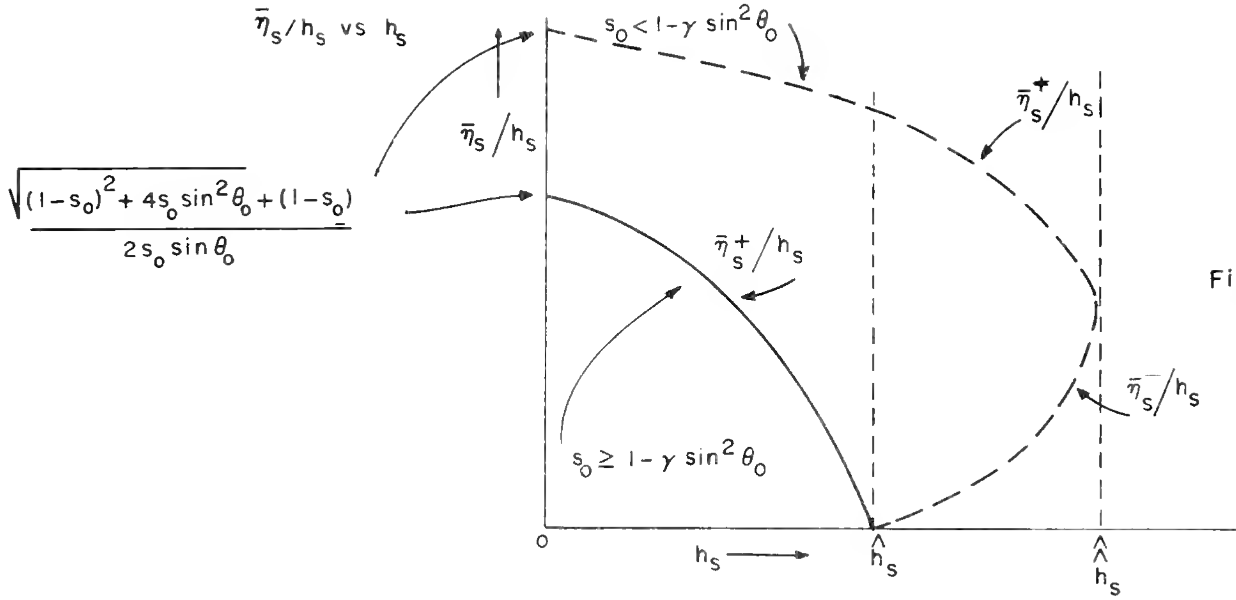

Figure 10(a)

$$\hat{h}_s = 2 \sin \theta_o$$

$$\hat{\hat{h}}_s = \frac{2\cos\theta_o \sqrt{(\gamma-1)(1-s_o)^2 + \gamma^2 s_o \sin\theta_o} - (2-\gamma)(1+s_o)\sin\theta_o}{2(\gamma-1) - \gamma^2 \sin^2\theta_o/2}$$

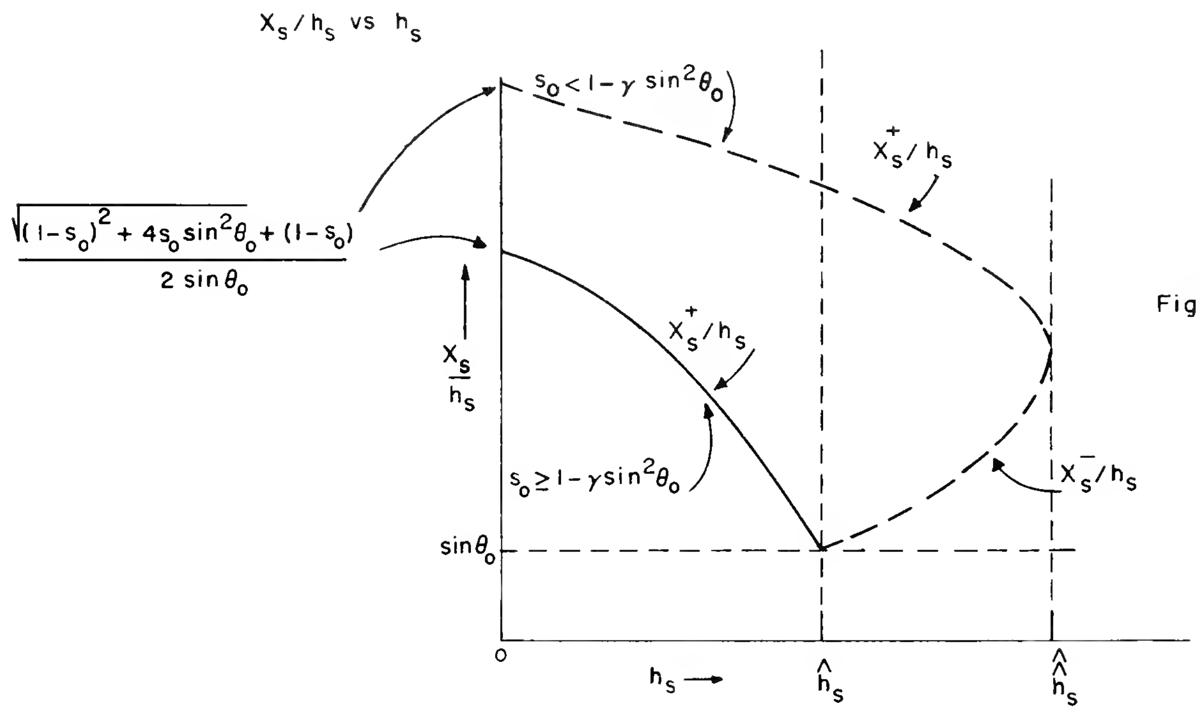

Figure 10(b)

Figures 10(a), 10(b). Sketch of the variation of $\bar{\eta}_s/h_s$ with h_s (a), and of X_s/h_s with h_s (b), in M_s shocks. The ($\pm$) superscripts on $\bar{\eta}_s$ and X_s indicate the plus and minus branches. The solid line curves are curves of type 1; the broken line curves are of type 2. X_s^+/h_s and $\bar{\eta}_s^+/h_s$, whatever the type, vary in a sense opposite to h_s; X_s^-/h_s and $\bar{\eta}_s^-/h_s$ vary in the same sense as h_s.

as $\bar{\eta}_s^+/h_s$. The solid curve of Figure 10(b) illustrates a typical (X_s^+/h_s) vs. h_s-curve. In contrast to the behavior of X_f^+/h_f, X_s^+/h_s is bounded for all allowable values of h_s.

6.2.2 $M_s^{(2)}$ curves $(s_o < 1-\gamma \sin^2\theta_o)$

M_2 shocks of type 2 exist only in the range

$$(6.12) \qquad 0 < h_s \leq \hat{\hat{h}}_s = \frac{-\sin\theta_o(2-\gamma)(1+s_o) + 2\cos\theta_o\sqrt{(\gamma-1)(1-s_o)^2 + s_o\gamma^2\sin^2\theta_o}}{2(\gamma-1) - (\gamma^2\sin^2\theta_o/2)}$$

where $\hat{\hat{h}}_s$ is a zero of $\hat{R}(h_s)$. In the range

$$(6.13) \qquad 0 < h_s \leq \hat{h}_s = 2\sin\theta_o,$$

only the plus branch $\bar{\eta}_s^+/h_s$ is admissible. In the range

$$(6.14) \qquad \hat{h}_s < h_s \leq \hat{\hat{h}}_s$$

both plus and minus branches are admissible. On the plus branch the values of $\bar{\eta}_s^+/h_s$ exceed those on the minus branch except at $h_s = \hat{\hat{h}}_s$ where the branches join smoothly. As h_s increases from zero to $\hat{\hat{h}}_s$, $\bar{\eta}_s^+/h_s$ decreases monotonically from the value $\bar{\eta}_s^+/h_s\big|_{h_s=0}$ given in (6.11) to $(\bar{\eta}_s^+/h_s)\big|_{h_s=\hat{\hat{h}}_s}$. As h_s decreases from $\hat{\hat{h}}_s$ to $\hat{h}_s$, $\bar{\eta}_s^-/h_s$ decreases monotonically from its maximum value $(\bar{\eta}_s^-/h_s\big|_{h_s=\hat{\hat{h}}_s}$ to zero. The general features of the (X_s/h_s) vs. h_s-curves are the same as those of the $(\bar{\eta}_s/h_s)$ vs. h_s-curves because of (6.7). Again (X_s/h_s) is bounded for all allowable valuesof h_s. The broken line curves of Figure 10(a) and 10(b) illustrate the general behavior of $\bar{\eta}_s/h_s$ and X_s/h_s in the allowable range of h_s.

6.3 Concluding remarks

It should be clear at this point that our analysis of M_s shocks may be made to parallel closely that of M_f shocks. It is a fact that almost every M_f result has a parallel M_s result. Evidently there is no need to give proofs of M_s results where this is the case. Some M_s results, however, have no parallel in M_f results. Of these the only one which does not follow more or less directly from the results of the preceding two subsections, and which, we believe, merits special attention, is the statement that $[S]_s$, m_s, $V^s_{n,o}$, <u>when regarded as functions of</u> h_s, <u>reach their maxima at the same value of</u> h_s <u>and that at this point,</u> $V^s_{n,1} = c_{s,1}$.*

To prove this statement it clearly suffices to show that $V^s_{n,1} = c_{s,1}$ at that value of h_s where m_s attains a maximum; for, $m_s = \rho_o V^s_{n,o}$ and, as we have seen [see subsection 3.2], m_s varies in the same sense as $[S]_s$.

Now following the procedure employed in subsection (5.5), it is easy to verify that

$$(6.15) \qquad 2(V^s_{n,1})^2 = (a_1^2 + \mu H_1^2 \rho_1^{-1}) - \sqrt{(a_1^2 + \mu H_1^2 \rho_1^{-1})^2 - 4b^2_{n,1} a_1^2 - 4(V^2_{n,1} - b^2_{n,1}) \frac{dm_s^2}{d\rho_1} G_1}$$

where the function G_1 is given in (5.59). When $dm_s^2/d\rho_1$ vanishes we find, using (1.4), (1.5), (6.15) and the fact that G_1 is here finite, that

$$(6.16) \qquad V_{n,1} = c_{s,1}.$$

To complete the proof it is clearly sufficient to prove 1) that

$$\frac{dm_s}{d\rho_1} = \frac{dm_s}{dh_s} \frac{dh_s}{d\rho_1}$$

vanishes when dm_s/dh_s vanishes and 2) that m_s regarded as a function of h_s has at least one stationary value.

---

*There is only one value of h_s for which this statement is true; but as we have mentioned in the footnote on p. 22 the proof is too lengthy to be included in this report.

Statement 1) will be proved if it can be shown that dm_s/dh_s and $d\rho_1/dh_s$ do not vanish simultaneously. Now it is easily verified that*

$$(6.17) \quad \frac{1}{\mu H_n^2} \frac{d(m_s^2/\rho_o)}{dh_s} = \frac{\left[1 - \sin\theta_o/h_s\right](d\overline{\eta}_s/dh_s) + \overline{\eta}_s\,(\overline{\eta}_s+1)\sin\theta_o/h_s^2}{\left[1+(\overline{\eta}/h_s)\sin\theta_o\right]^2} .$$

Since $\theta_o \neq 0$, $h_s > 0$ and $\overline{\eta}_s \geq 0$ it is clear that dm_s/dh_s and $d\overline{\eta}_s/dh_s$ cannot vanish simultaneously.

Statement 2) is an immediate consequence of Rolle's Theorem. Consider $M_s^{(1)}$ shocks for example. In this type of shock

$$(6.18) \quad \frac{m_s}{\rho_o} = \left[\frac{\eta_s\, b_{n,o}^2}{1+(\overline{\eta}_s/h_s)\sin\theta_o}\right]^{1/2} = \left[\frac{(\overline{\eta}_s+1)b_{n,o}^2}{1+(\overline{\eta}_s/h_s)\sin\theta_o}\right]^{1/2}$$

[cf. (6.3)] is a single-valued differentiable function of h_s, $0 \leq h_s \leq 2\sin\theta_o$, which assumes the value $b_{n,o}$ when $h_s = \sin\theta_o$ and when $h_s = 2\sin\theta_o$ (Note $\overline{\eta}_s = 0$ at $h_s = 2\sin\theta_o$); dm_s/dh_s therefore vanishes at some interior point of the interval $\sin\theta_o \leq h_s \leq 2\sin\theta_o$. Similar reasoning applied to $M_s^{(2)}$ shocks leads to the conclusion that dm_s/dh_s vanishes for at least one interior point of the interval, $\sin\theta_o \leq h_s \leq \hat{\hat{h}}_s$, where $\hat{\hat{h}}_s$ is given in (6.12). A more detailed analysis reveals that the vanishing dm_s/dh_s occurs at a value of h_s exceeding $1.5\sin\theta_o$.

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*This equation follows from (6.3) and the identity

$$\frac{d\overline{\eta}_s/h_s}{dh_s} = \frac{1}{h_s}\frac{d\overline{\eta}_s}{dh_s} - \frac{\overline{\eta}_s}{h_s^2} .$$

7. Limit shocks

7.1 Fast 0^o-limit shocks

7.1.1 Fast 0^o - limit shocks of type 1 ($s_o \geq 1$)

The analysis of this subsection is based on the identity

$$(7.1)\qquad \frac{d\bar{\eta}_f}{dh_f} = \frac{\bar{\eta}_f}{h_f} + h_f \frac{d(\bar{\eta}_f/h_f)}{dh_f}$$

and the equations [cf. (5.5) and (5.8)]

$$(7.2)\qquad h_f = \bar{\eta}_f \left\{ \frac{2\, s_o \sin\theta_o - (\gamma-1)h_f}{-\frac{\gamma}{2} h_f \sin\theta_o - (1-s_o) + \sqrt{R(h_f)}} \right\},$$

$$(7.2')\qquad R(h_f) = \left(\frac{h_f}{2}\right)^2 \left\{ \gamma^2 \sin^2\theta_o - 4(\gamma-1) \right\} + h_f \sin\theta_o (2-\gamma)(1+s_o)$$
$$+ (1-s_o)^2 + 4\, s_o \sin^2\theta_o,$$

where

$$(7.3)\qquad 0 < h_f \leq \frac{2\sin\theta_o}{\gamma-1},$$

and where only the plus sign appears before the radical in accordance with the fact that we are dealing with shocks of type 1.

Now let us suppose that $\bar{\eta}_f$, $0 < \bar{\eta}_f \leq 2/(\gamma-1)$, is fixed and let $h_f = h_f(\bar{\eta}_f, \theta_o)$ be the corresponding value of h_f. It follows immediately from (7.3) that

$$(7.4)\qquad \lim_{\theta_o \to 0} h_f(\bar{\eta}_f, \theta_o) = 0, \qquad 0 < \bar{\eta}_f \leq \frac{2}{\gamma-1},$$

If we take the limit of both sides of (7.2), we find that non-vanishing values of

$\bar{\eta}_f, 0 < \bar{\eta}_f \leq 2/(\gamma-1)$, are compatible with the vanishing of h_f in the $\theta_o = 0^o$ limit. Employing (7.1), (5.6) and (5.7) we find

$$(7.5) \qquad \lim_{\theta_o \to 0} \frac{dh_f}{d\bar{\eta}_f} = 0, \qquad 0 < \bar{\eta}_f \leq 2/(\gamma-1).$$

Thus, in the $(h_f, \bar{\eta}_f)$ plane the limiting process may be described as follows: As θ_o approaches zero the range of h_f shrinks and the $\bar{\eta}_f$ vs. h_f-curves become steeper. In the limit h_f vanishes and $\bar{\eta}_f$ becomes independent of h_f.

Taking the limit of both sides of equations $(4.5\text{-}B_1\text{-}B_4)$ we find, in virtue of (7.4), that

$$(7.6) \qquad \bar{Y}_f = \frac{2\,\bar{\eta}_f\,\gamma}{2-(\gamma-1)\bar{\eta}_f}, \qquad 0 < \bar{\eta}_f \leq 2/(\gamma-1),$$

$$(7.7) \qquad m_f^2 = -[p]_f/[\tau]_f,$$

$$(7.8) \qquad [u_n]_f = -m_f \tau_1^f \bar{\eta}_f,$$

$$(7.9) \qquad [u_y]_f = 0.$$

These equations will be recognized as the well-known gas dynamical jump relations; they lead directly to the fast gas shock equations, namely (2.45).

7.1.2 <u>Fast 0^o-limit shocks of type 2</u> ($s_o < 1$)

Consider the $\bar{\eta}_f$ vs. h_f curves of Figure 3(b). Note that $h_f = h_f(\bar{\eta}_f, \theta_o)$ approaches zero as θ_o approaches 0^o on the upper branch (minus branch) in some neighborhood of $\bar{\eta}_f = 2/(\gamma-1)$ while outside this neighborhood h_f approaches a finite value depending on $\bar{\eta}_f$ [see 0^o-curve of Figure 3(b)].

These observations may be verified analytically as follows. Let us take the limit of both sides of

$$(7.10) \qquad \frac{\bar{\eta}_f}{h_f} = \frac{\left[-\frac{\gamma}{2} h_f \sin\theta_o - (1-s_o) \pm \sqrt{R_\chi(h_f)}\right]}{2s_o \sin\theta_o - (\gamma-1)\, h_f},$$

where

$$(7.11) \qquad 0 < h_f \leq \hat{\hat{h}}_f(\theta_o),$$

[See (5.5) and (5.23)] without assuming that h_f also approaches zero. We then find that

$$(7.12) \qquad \bar{\eta}_f = \frac{(1-s_o) \pm \sqrt{(1-s_o)^2 - (\gamma-1)h_f^2}}{(\gamma-1)}$$

where

$$(7.13) \qquad 0 < h_f \leq (1-s_o)/\sqrt{(\gamma-1)}.$$

Equation (7.12), when solved for h_f^2, becomes

$$(7.14) \qquad h_f^2 = \bar{\eta}_f \left\{ 2(1-s_o) - (\gamma-1)\bar{\eta}_f \right\}.$$

The condition $h_f^2 > 0$ implies

$$(7.15) \qquad 0 < \bar{\eta}_f \leq \frac{2(1-s_o)}{\gamma-1} \equiv \bar{\eta}_{f,\text{crit.}} < 2/(\gamma-1).$$

Assuming that $\bar{\eta}_f$ is confined to this range, and taking the limits of both sides of (5.1), (5.2), (5.3), (5.4) we obtain a set of equations which lead directly to those listed in (2.5.1) - that is, to the equations which determine the state behind an 'incomplete' switch-on shock. We now ask what is the limit solution corresponding to values of $\bar{\eta}_f$ in the range

(7.16) $\quad \bar{\eta}_{f,crit} < \bar{\eta}_f \leq 2/(\gamma-1).$

To determine this solution we expand (7.10) in the neighborhood of $h_f = 2 \sin \theta_o/(\gamma-1)$ and assume that h_f is small when θ_o is small. The result is

(7.17) $$\bar{\eta}_f = \frac{2(1-s_o)}{(\gamma-1)-2\, s_o \sin \theta_o/h_f} + \ldots ,$$

where the dots indicate terms of higher order in h_f and $\sin \theta_o$. From (7.17) it follows that

(7.18) $$\bar{\eta}_f \geq \frac{2(1-s_o)}{\gamma-1} = \bar{\eta}_{f,crit}$$

for any small value of h_f and $\sin \theta_o$. Solving for h_f, we find

(7.19) $$h_f = \frac{2 s_o \sin \theta_o}{(\gamma-1)(\bar{\eta}_f - \bar{\eta}_{f,crit})}\bar{\eta}_f + \ldots .$$

From this equation it follows that

(7.20) $$\lim_{\theta_o \to 0} h_f = \lim_{\theta_o \to 0} h_f(\bar{\eta}_f, \theta_o) = 0, \quad \bar{\eta}_{f,crit} < \bar{\eta}_f \leq \frac{2}{\gamma-1} .$$

By following the procedure employed in subsection (7.1), it is now an easy matter to verify that the limit solution is an 'incomplete' fast gas shock [see Subsection 2.6.2]. It is also a simple matter to verify that the state behind the 'incomplete' switch-on shock passes continuously into the state behind the incomplete gas shock as $\bar{\eta}_f$ passes through the critical value, $\bar{\eta}_{crit}$.

7.2 Fast 90°-limit shocks ($s_o \geq 1$)

We have seen that 90°-limit shocks are all of type 1. If we take the limit as θ_o approaches 90° of both sides of the last equations in (5.4) and (5.5) we find, using the fact that only the plus branch is admissible in type 1 shocks and the fact that $s_o \geq 1$,

$$(7.21) \qquad \bar{\eta}_f/h_f = 1.$$

This equation together with the limit equations resulting from (5.1) - (5.3) and the Hugoniot relation lead directly to the equations listed in (2.65) - that is, to the equations which determine the state behind fast perpendicular shocks.

7.3 Slow limit shocks

The various slow limit shocks described in subsections 2.6.3 and 2.6.5 may be obtained as limits of (6.1) - (6.5) or (4.5) B_1 - B_4 by employing methods similar to those used above; a detailed derivation will therefore be omitted.

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